Find And Classify The Critical Points Of Z = (x^2 6x) (y^2 4y). Local Maximums: _____Local Minimums:

Find And Classify The Critical Points Of Z = (x^2 + 6x) (y^2 + 4y). Local Maximums: _____Local Minimums:

Understanding how to find and classify critical points of a multivariable function is a fundamental concept in calculus, especially in the context of optimization problems. The function in question, Z = (x^2 + 6x)(y^2 + 4y), presents a rich case study for applying techniques such as setting derivatives to zero, analyzing second derivatives, and understanding the nature of critical points. This article offers a comprehensive guide on how to find and classify the critical points of this function, detailing each step and providing insights into the process of determining local maximums and minimums.

Introduction to Critical Points and Their Classification

In multivariable calculus, critical points are points in the domain where the gradient vector of the function vanishes—that is, where all first-order partial derivatives are zero. These points are essential because they can be locations of local maxima, local minima, or saddle points.

Key Concepts:

- Critical Point: A point (x, y) where ∇Z = 0.
- Second Derivative Test: Used to classify the nature of a critical point based on second derivatives.

Why Classify Critical Points?

Classifying critical points helps in understanding the behavior of the function around those points, which is particularly useful in optimization problems, physics, economics, and engineering.

Step 1: Expressing the Function Clearly

Before proceeding with derivatives, it's important to write the function clearly:

\[Z = (x^2 + 6x)(y^2 + 4y) \]

This is a product of two quadratic expressions, which suggests that the critical points can be found by analyzing the derivatives with respect to x and y separately.

Step 2: Computing Partial Derivatives

To find the critical points, we need the first-order partial derivatives of Z concerning x and y.

Partial derivative with respect to x:

Using the product rule:

$$
\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x} (x^2 + 6x)(y^2 + 4y)
= (2x + 6)(y^2 + 4y)
$$

Similarly, the partial derivative with respect to y:

Partial derivative with respect to y:

$$
\frac{\partial Z}{\partial y} = (x^2 + 6x) \frac{\partial}{\partial y}(y^2 + 4y) = (x^2 + 6x)(2y + 4)
$$

Step 3: Setting Partial Derivatives Equal to Zero

Critical points occur where both partial derivatives are zero simultaneously:

$$
(2x + 6)(y^2 + 4y) = 0
$$
$$
(x^2 + 6x)(2y + 4) = 0
$$

These products equal zero when either factor in each is zero, leading to four cases.

Case 1:

\[

$$
2x + 6 = 0 \quad \Rightarrow \quad x = -3
$$

$$
y^2 + 4y = 0 \quad \Rightarrow \quad y(y + 4) = 0 \quad \Rightarrow \quad y=0 \quad \text{or} \quad y=-4
$$

Critical points here:

- $(-3, 0)$
- $(-3, -4)$

Case 2:

$$
x^2 + 6x = 0 \quad \Rightarrow \quad x(x + 6) = 0 \quad \Rightarrow \quad x=0 \quad \text{or} \quad x=-6
$$

$$
2y + 4 = 0 \quad \Rightarrow \quad y = -2
$$

Critical points:

- $(0, -2)$
- $(-6, -2)$

Step 4: List of Critical Points

Summarizing, the critical points are:

1. $(-3, 0)$
2. $(-3, -4)$
3. $(0, -2)$
4. $(-6, -2)$

Step 5: Classifying Critical Points Using the Second Derivative Test

To classify these critical points as local maxima, minima, or saddle points, we analyze the second derivatives and compute the discriminant D:

$$
D = Z_{xx} Z_{yy} - (Z_{xy})^2

\]

Where:

- \(Z_{xx} \) = second partial derivative with respect to x,
- \(Z_{yy} \) = second partial derivative with respect to y,
- \(Z_{xy} \) = mixed second partial derivative.

Calculating Second Derivatives:

From the first derivatives:

\[
Z_x = (2x + 6)(y^2 + 4y)
\]
\[
Z_y = (x^2 + 6x)(2y + 4)
\]

Compute second derivatives:

\[
Z_{xx} = \frac{\partial}{\partial x} Z_x = \frac{\partial}{\partial x} [(2x + 6)(y^2 + 4y)] = 2(y^2 + 4y)
\]
\[
Z_{yy} = \frac{\partial}{\partial y} Z_y = \frac{\partial}{\partial y} [(x^2 + 6x)(2y + 4)] = 2(x^2 + 6x)
\]
\[
Z_{xy} = \frac{\partial}{\partial y} Z_x = \frac{\partial}{\partial y} [(2x + 6)(y^2 + 4y)] = (2x + 6)(2y + 4)
\]

Now, evaluate the second derivatives at each critical point.

Step 6: Evaluating and Classifying Each Critical Point

Let's evaluate \(Z_{xx} \), \(Z_{yy} \), and \(Z_{xy} \) at each critical point, then compute \(D \):

Critical Point 1: \((-3, 0)\)

- $Z_{xx} = 2(0^2 + 4 \times 0) = 0$
- $Z_{yy} = 2((-3)^2 + 6 \times -3) = 2(9 - 18) = 2(-9) = -18$
- $Z_{xy} = (2 \times -3 + 6)(2 \times 0 + 4) = (-6 + 6)(4) = 0 \times 4 = 0$

Discriminant:

$$
D = (0)(-18) - (0)^2 = 0
$$

Since D=0, the second derivative test is inconclusive. Further analysis is needed, but initial indicators suggest a saddle point or a flat point.

Critical Point 2: $(-3, -4)$

- $Z_{xx} = 2((-4)^2 + 4 \times -4) = 2(16 - 16) = 2(0) = 0$
- $Z_{yy} = 2(9 + 6 \times -3) = 2(9 - 18) = 2(-9) = -18$
- $Z_{xy} = (-6 + 6)(2 \times -4 + 4) = 0 \times (-8 + 4) = 0$

Discriminant:

$$
D = 0 \times -18 - 0^2 = 0
$$

Again, D=0, indicating an inconclusive classification.

Critical Point 3: $(0, -2)$

- $Z_{xx} = 2((-2)^2 + 4 \times -2) = 2(4 - 8) = 2(-4) = -8$
- $Z_{yy} = 2(0^2 + 6 \times 0) = 0$
- $Z_{xy} = (2 \times 0 + 6)(2 \times -2 + 4) = 6 \times (-4 + 4) = 6 \times 0 = 0$

Discriminant:

$$
D = (-8)(0) - 0^2 = 0
$$

Inconclusive again.

Critical Point 4: $(-6, -2)$

- $Z_{xx} = 2((-2)^2 + 4 \times -2) = 2(4 - 8) = 2(-4) = -8$
- $Z_{yy} = 2((-6)^2 + 6 \times -6) = 2(36 - 36) = 2(0) = 0$

- \(Z_{xy}

Frequently Asked Questions

How do you find the critical points of the function Z = (x^2 + 6x)(y^2 + 4y)?

To find the critical points, first compute the partial derivatives of Z with respect to x and y, set them equal to zero, and solve the resulting system of equations.

What are the steps to classify the critical points as local maxima, minima, or saddle points for Z = (x^2 + 6x)(y^2 + 4y)?

After finding the critical points, compute the second derivatives to form the Hessian matrix. Then, analyze the Hessian's definiteness at each critical point: positive definite indicates a local minimum, negative definite indicates a local maximum, and indefinite indicates a saddle point.

What are the critical points of Z = (x^2 + 6x)(y^2 + 4y)?

The critical points are found by solving the system derived from the first derivatives. Specifically, critical points occur where either x = -3 or x = 0, and y = -2 or y = 0, leading to points (-3, -2), (-3, 0), (0, -2), and (0, 0).

How do you determine whether the critical points are local maximums or minimums for this function?

Evaluate the second derivatives at each critical point and analyze the Hessian matrix. If the Hessian is positive definite at a point, it's a local minimum; if negative definite, it's a local maximum.

What are the identified local maximums and minimums for Z = (x^2 + 6x)(y^2 + 4y)?

Based on the second derivative test, the points (-3, -2) and (0, 0) are local maximums, while (-3, 0) and (0, -2) are local minimums.

Why is classifying critical points important in analyzing the function Z = (x^2 + 6x)(y^2 + 4y)?

Classifying critical points helps understand the function's local behavior, such as where it reaches high or low values, which is essential in optimization problems and understanding the function's overall shape.

Additional Resources

Find And Classify The Critical Points Of Z = (x^2 6x) (y^2 4y)

Understanding the nature of critical points in multivariable functions is fundamental in calculus, especially when analyzing the behavior of functions across different domains. The function

$$Z = (x^2 + 6x) (y^2 + 4y)$$

serves as an intriguing case study for such analysis. Its structure, composed of two quadratic factors, hints at a rich landscape of local maxima, minima, and saddle points. This article aims to comprehensively explore and classify the critical points of this function, providing insight into the underlying mathematical mechanics and offering a detailed analytical framework for students, educators, and enthusiasts alike.

Understanding the Function Structure

Before delving into the critical points, it is essential to understand the structure of the function:

$$Z(x, y) = (x^2 + 6x)(y^2 + 4y)$$

This is a product of two quadratic expressions, each in a different variable. The function’s behavior depends on the interplay between these factors, which can be positive, negative, or zero depending on the values of x and y.

Key observations:

- The function is defined for all real x and y.
- The factors $x^2 + 6x$ and $y^2 + 4y$ are quadratic, opening upwards, with their respective minima and roots.
- The critical points occur where the first derivatives vanish, which involves analyzing the partial derivatives with respect to both variables.

Finding Critical Points

Critical points are points where the gradient of the function, ∇Z, equals zero. For the function $Z(x, y)$, this involves solving the system:

$$
\frac{\partial Z}{\partial x} = 0, \quad \frac{\partial Z}{\partial y} = 0
$$

Step 1: Compute Partial Derivatives

Given:

$$
Z = (x^2 + 6x)(y^2 + 4y)
$$

Apply the product rule:

$$
\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x} (x^2 + 6x) \times (y^2 + 4y) + (x^2 + 6x) \times \frac{\partial}{\partial x} (y^2 + 4y)
$$

Since $y^2 + 4y$ does not depend on x:

$$
\frac{\partial Z}{\partial x} = (2x + 6)(y^2 + 4y)
$$

Similarly, for $\frac{\partial Z}{\partial y}$:

$$
\frac{\partial Z}{\partial y} = (x^2 + 6x) \times (2y + 4)
$$

Step 2: Set Partial Derivatives to Zero

Critical points satisfy:

$$
(2x + 6)(y^2 + 4y) = 0 \quad \text{(1)}
$$

$$
(x^2 + 6x)(2y + 4) = 0 \quad \text{(2)}
$$

Solving for Critical Points

The system reduces to solving the following cases based on each factor being zero:

Case 1: $2x + 6 = 0$

$$
x = -3
$$

Substitute into (1):

$$
(2(-3) + 6)(y^2 + 4y) = 0 \implies (-6 + 6) (y^2 + 4 y) = 0 \implies 0 \times (y^2 + 4 y) = 0
$$

This is always true, so for $x = -3$, the critical points occur at all y satisfying:

$$
(x^2 + 6x)(2 y + 4) = 0
$$

Now, from (2):

$$
(x^2 + 6x)(2 y + 4) = 0
$$

At $x = -3$:

$$
((-3)^2 + 6(-3))(2 y + 4) = (9 - 18)(2 y + 4) = (-9)(2 y + 4) = 0
$$

This implies:

$$
2 y + 4 = 0 \Rightarrow y = -2
$$

Thus, critical point:

$$
\boxed{(-3, -2)}
$$

Case 2: $y^2 + 4 y = 0$

$$
y(y + 4) = 0 \Rightarrow y = 0 \quad \text{or} \quad y = -4
$$

At these \(y\)-values, from (2):

\[
(x^2 + 6x)(2 y + 4) = 0
\]

- For \(y = 0\):

\[
(x^2 + 6x)(2 \times 0 + 4) = (x^2 + 6x) \times 4 = 0
\]

Thus:

\[
x^2 + 6x = 0 \Rightarrow x(x + 6) = 0 \Rightarrow x = 0 \quad \text{or} \quad x = -6
\]

Critical points:

\[
(0, 0) \quad \text{and} \quad (-6, 0)
\]

- For \(y = -4\):

\[
(x^2 + 6x)(2 \times (-4) + 4) = (x^2 + 6x)(-8 + 4) = (x^2 + 6x)(-4) = 0
\]

So:

\[
x^2 + 6x = 0 \Rightarrow x = 0 \quad \text{or} \quad x = -6
\]

Critical points:

\[
(0, -4) \quad \text{and} \quad (-6, -4)
\]

Summary of Critical Points:

\[
\boxed{
\begin{aligned}
&(-3, -2) \\
&(0, 0) \\
&(-6, 0) \\
&(0, -4) \\

&(-6, -4)
\end{aligned}
}
\]

Classifying Critical Points: Maxima, Minima, or Saddle Points

Having identified the critical points, the next step involves classification. This process employs the second derivative test for functions of two variables, which uses the second partial derivatives to determine the nature of each critical point.

Step 1: Compute Second Partial Derivatives

Recall:

\[
Z_x = (2x + 6)(y^2 + 4 y)
\]
\[
Z_y = (x^2 + 6x)(2 y + 4)
\]

Second derivatives:

\[
Z_{xx} = \frac{\partial}{\partial x} Z_x = \frac{\partial}{\partial x} \left[(2x + 6)(y^2 + 4 y) \right] = 2 (y^2 + 4 y)
\]

\[
Z_{yy} = \frac{\partial}{\partial y} Z_y = \frac{\partial}{\partial y} \left[(x^2 + 6x)(2 y + 4) \right] = (x^2 + 6x) \times 2
\]

\[
Z_{xy} = Z_{yx} = \frac{\partial}{\partial y} Z_x = \frac{\partial}{\partial y} \left[(2x + 6)(y^2 + 4 y) \right] = (2x + 6)(2 y + 4)
\]

Step 2: Evaluate the Second Derivatives at Critical Points

The second derivative test involves calculating the discriminant:

\[
D = Z_{xx} Z_{yy} - (Z_{xy})^2

\]

- If \(D > 0\) and \(Z_{xx} > 0\), the point is a local minimum.
- If \(D > 0\) and \(Z_{xx} < 0\), the point is a local maximum.
- If \(D < 0\), the point is a saddle point.
- If \(D = 0\), the test is inconclusive.

Analyzing Each Critical Point

1. Critical point \((-3, -2)\)

Calculate derivatives:

\[
Z_{xx} = 2(y^2 + 4 y) = 2((-2)^2 + 4 \times -2) = 2(4 - 8) = 2(-4) = -8
\]

\[
Z_{yy} = 2(x^2 + 6x) = 2(9 - 18) = 2(-9) = -18

Find And Classify The Critical Points Of Z X 2 6x Y 2 4y Local Maximums Local Minimums

Find And Classify The Critical Points Of Z X 2 6x Y 2 4y Local Maximums Local Minimums

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