Find Parametric Equations For The Line Through (5, 4, 4) That Is Perpendicular To The Plane X Y + 2z

Find Parametric Equations For The Line Through (5, 4, 4) That Is Perpendicular To The Plane X Y + 2z is a classic problem in analytic geometry that involves understanding the relationship between a plane and a line perpendicular to it. Such problems often appear in calculus and linear algebra courses, as they test your ability to interpret geometric concepts algebraically. In this article, we will explore how to derive the parametric equations of a line passing through a given point and perpendicular to a specified plane, step by step.

Understanding the Problem

Before diving into the solution, it's essential to comprehend what the problem asks:

- Point on the line: (5, 4, 4)
- Plane equation: XY + 2z = 0
- Line property: Perpendicular to the plane

The goal is to find the parametric equations of the line that passes through the point (5, 4, 4) and is perpendicular to the plane described by XY + 2z = 0.

Key Concepts and Definitions

Planes and Their Normal Vectors

A plane in three-dimensional space can typically be written in the form:

\[ax + by + cz + d = 0 \]

where the vector \(\vec{n} = \langle a, b, c \rangle \) is the normal vector to the plane. The normal vector is perpendicular to every line lying within the plane.

In our case, the plane equation is:

\[XY + 2z = 0 \]

which involves a product term \(XY \), making it nonlinear. Therefore, this is not a standard linear plane; it's a quadratic surface. However, the problem asks for a line perpendicular to the surface at some point, which suggests a different interpretation.

Important clarification: In typical geometry problems, when asked for a line perpendicular to a plane, the plane is usually expressed in linear form.

Here, the equation involves a product XY, which indicates a surface (a hyperbolic paraboloid or similar). But, considering the context, it is likely a typo or a simplification, and the intended plane is:

$$ x y + 2z = 0 $$

which is a linear plane.

Assumption: The plane equation is $xy + 2z = 0$, a linear surface, and we seek a line passing through (5, 4, 4) perpendicular to it.

If the original problem truly involves a nonlinear surface, then the concept of a line "perpendicular" to it is more complex, involving the gradient vector at a point. For this article, we will proceed assuming the plane is:

$$ xy + 2z = 0 $$

which is linear, and the problem makes sense in the context of finding a line perpendicular to this plane.

Finding the Normal Vector to the Plane

Step 1: Rewrite the plane in standard form

Given:

$$ xy + 2z = 0 $$

this is a surface, not a plane. To find a line perpendicular to it at a point, we need the gradient vector of the surface at that point, which acts as the normal vector.

Step 2: Compute the gradient vector

The surface is given by:

$$ F(x, y, z) = xy + 2z $$

The gradient vector ∇F is:

$$ \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) $$

Calculating each partial derivative:

- $\frac{\partial F}{\partial x} = y$
- $\frac{\partial F}{\partial y} = x$
- $\frac{\partial F}{\partial z} = 2$

Thus,

$$ \nabla F = \langle y, x, 2 \rangle $$

At the point $(x, y, z) = (5, 4, 4)$:

$$ \nabla F = \langle 4, 5, 2 \rangle $$

This gradient vector is perpendicular to the surface at the point $(5, 4, 4)$. Therefore, the line perpendicular to the surface at this point has direction vector:

$$ \vec{d} = \langle 4, 5, 2 \rangle $$

Note: If the original problem intended for the plane to be linear, or if the surface's normal at the point is given by the gradient, then this approach applies.

Deriving the Parametric Equations of the Line

Step 1: Identify a point on the line

The problem states that the line passes through the point:

$$ P_0 = (5, 4, 4) $$

which is on the line.

Step 2: Use the direction vector

The direction vector $\vec{d}$ for the line is:

$$ \vec{d} = \langle 4, 5, 2 \rangle $$

This vector points in the direction perpendicular to the surface at the specified point, ensuring the line is perpendicular to the surface.

Step 3: Write the parametric equations

The parametric equations of a line passing through point P_0 with direction vector $\vec{d}$ are:

$$
\begin{cases}
x = x_0 + t \cdot d_x \\
y = y_0 + t \cdot d_y \\
z = z_0 + t \cdot d_z
\end{cases}
$$

where t is a parameter in $\mathbb{R}$.

Plugging in the known values:

$$
\boxed{
\begin{cases}
x = 5 + 4t \\
y = 4 + 5t \\
z = 4 + 2t
\end{cases}
}
$$

This set of equations describes the line passing through (5, 4, 4) and perpendicular to the surface $xy + 2z = 0$ at that point.

Summary of the Solution

- The key to solving such problems is recognizing whether the surface is linear or nonlinear.
- For a surface defined by $xy + 2z = 0$, the gradient vector at a point provides the normal direction.
- The line perpendicular to the surface at a point has the same direction as the gradient vector.
- The parametric equations are derived using the point and the direction vector.

Additional Considerations

What if the plane was linear?

If the plane equation had been:

$$ a x + b y + c z = d $$

then the normal vector would be simply $\langle a, b, c \rangle$. The process would be similar: use the point and the normal vector to write the parametric equations.

Dealing with Nonlinear Surfaces

When the surface is nonlinear (like $xy + 2z = 0$), the gradient vector at a point indicates the direction of the steepest ascent and is perpendicular to the tangent plane at that point. To find a line perpendicular to the surface, the line should follow this gradient vector.

Conclusion

Finding the parametric equations of a line passing through a specific point and perpendicular to a surface involves understanding both the geometric and algebraic properties of the surface. In the context of the surface $xy + 2z = 0$, the key steps involve computing the gradient vector at the point of interest and then using it as the direction vector for the line. The final parametric equations:

$$
x = 5 + 4t, \quad y = 4 + 5t, \quad z = 4 + 2t
$$

provide a complete description of the line. These equations are invaluable in applications ranging from computer graphics to physics, where understanding the orientation and position of lines relative to surfaces is essential.

Frequently Asked Questions

How do you find the parametric equations of a line passing through the point (5, 4, 4) and perpendicular to the plane XY + 2z?

First, identify the normal vector of the plane, which is (1, 1, 2) for XY + 2z. Since the line is perpendicular to the plane, its direction vector is the same as the normal vector. Then, use the point (5, 4, 4) and the direction vector to write parametric equations: $x = 5 + t$, $y = 4 + t$, $z = 4 + 2t$.

What is the normal vector of the plane XY + 2z?

The normal vector of the plane XY + 2z is (1, 1, 2), derived from the coefficients of x, y, and z in the plane equation.

How do I verify that the line is perpendicular to the plane XY + 2z?

You verify by confirming that the direction vector of the line is parallel to the plane's normal vector. Since the line's direction vector is (1, 1, 2), which matches the plane's normal vector, the line is perpendicular to the plane.

Can the parametric equations be written differently for the same line?

Yes, you can write the parametric equations using different parameters or rearranged forms, but the core form $x = 5 + t$, $y = 4 + t$, $z = 4 + 2t$ remains standard for this problem.

What is the significance of the point (5, 4, 4) in

this context?

The point (5, 4, 4) is the point through which the line passes. It serves as the initial point in the parametric equations to define the line's position in space.

How does the equation XY + 2z = 0 relate to the line's parametric equations?

The plane equation XY + 2z = 0 defines the plane perpendicular to the line. The line's parametric equations describe a line that intersects this plane at a right angle, passing through (5, 4, 4).

What are the steps to derive parametric equations for a line given a point and a normal vector?

1. Identify the point through which the line passes. 2. Use the normal vector of the plane as the direction vector of the line. 3. Write the parametric equations as $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, where (x_0, y_0, z_0) is the point and (a, b, c) is the direction vector.

Additional Resources

Parametric Equations for a Line Perpendicular to a Plane: An Expert Breakdown

When navigating the fascinating world of three-dimensional geometry, one of the fundamental tasks involves understanding how lines and planes interact within space. Whether you're a student grappling with multivariable calculus or a seasoned professional applying geometric principles to engineering or computer graphics, mastering the derivation of parametric equations for lines in 3D is essential. Today, we'll explore a specific, yet common, problem: finding the parametric equations for a line passing through the point (5, 4, 4) that is perpendicular to the plane defined by \(xy + 2z = 0 \).

This task might seem straightforward at first glance, but it involves a nuanced understanding of the relationships between points, lines, and planes in three-dimensional space. Let's delve into this problem with a detailed, step-by-step approach, dissecting each component with expert precision.

Understanding the Core Concepts

Before diving into the solution, it's crucial to review some fundamental concepts in 3D geometry:

1. Planes in 3D Space
A plane in three dimensions can be expressed in the general form:

\[A x + B y + C z + D = 0 \]

where \((A, B, C) \) is the normal vector to the plane, and \(D \) is a scalar constant.

2. Normal Vectors
The normal vector $\vec{n} = (A, B, C)$ is perpendicular to the plane. It indicates the direction in which the plane faces, and it is vital for defining perpendicular relationships.

3. Lines in 3D Space
A line can be represented parametrically as:

$$
\begin{cases}
x = x_0 + t a \\
y = y_0 + t b \\
z = z_0 + t c
\end{cases}
$$

where (x_0, y_0, z_0) is a point on the line, (a, b, c) is the direction vector of the line, and t is a parameter.

4. Perpendicularity Condition
For a line to be perpendicular (orthogonal) to a plane, its direction vector must be parallel to the plane's normal vector. This is a key property: the direction vector of the line is aligned with the normal vector of the plane.

Step-by-Step Solution Approach

Given the specific problem, let's now formalize the approach:

Step 1: Identify the Plane's Normal Vector

The plane is given by:

$$
xy + 2z = 0
$$

This is a quadratic form involving xy, which complicates direct identification of the normal vector via the standard form. To proceed, rewrite the plane equation in a form that highlights the normal vector.

However, the equation:

$$
xy + 2z = 0
$$

is not explicitly linear in x, y, and z. To find the normal vector, we need to interpret the geometric meaning.

Key insight: The normal vector to a plane is derived from the coefficients of the linear form in x, y, z. But here, the equation is quadratic in x and y. This suggests an initial misinterpretation: the equation $xy + 2z = 0$ defines a surface, not a plane.

But the problem explicitly states "the plane $xy + 2z$", which hints at a typo or an abbreviation. Likely, the intended plane is:

$$
x + y + 2z = 0
$$

which is linear and standard for such problems.

Assumption: For this problem to be consistent with the question's intent, we'll assume the plane is:

$$
x + y + 2z = 0
$$

This is a common form, and its normal vector is straightforward to extract.

Step 2: Extract the Normal Vector

From the assumed plane:

$$
x + y + 2z = 0
$$

The coefficients of x, y, and z give the normal vector:

$$
\boxed{\vec{n} = (1, 1, 2)}
$$

This vector is perpendicular to the plane and will serve as the direction vector for our line.

Note: If the original plane truly was $xy + 2z = 0$, the problem becomes more complex, involving surface normals that depend on the gradient of the function. But given the canonical form and typical problem structure, the assumption above is reasonable.

Step 3: Determine the Direction Vector of the Line

Since the line must be perpendicular to the plane, and the plane's normal vector is $\vec{n} = (1, 1, 2)$, the direction vector $\vec{d}$ of the line is parallel to $\vec{n}$:

$$
\boxed{\vec{d} = (1, 1, 2)}
$$

This aligns with the geometric principle that the shortest path from a point to a plane is along a line perpendicular to the plane.

Step 4: Establish a Point on the Line

The line must pass through the point $P = (5, 4, 4)$. This is straightforward: the line's parametric equations will be based on this point.

Step 5: Write the Parametric Equations

Using the point $(5, 4, 4)$ and the direction vector $(1, 1, 2)$, the parametric equations are:

$$
\begin{cases}
x(t) = 5 + t \\
y(t) = 4 + t \\
z(t) = 4 + 2t
\end{cases}
$$

where $t \in \mathbb{R}$.

Final Parametric Equations and Interpretation

The parametric equations describing the line through $(5, 4, 4)$ and perpendicular to the plane $x + y + 2z = 0$ are:

$$
\boxed{
\begin{aligned}
x(t) &= 5 + t \\
y(t) &= 4 + t \\
z(t) &= 4 + 2t
\end{aligned}
}
$$

Key Features:

- The line passes through the given point when $t = 0$.
- The direction vector $(1, 1, 2)$ ensures perpendicularity with the plane.
- The parametric form allows for easy plotting, analysis, and intersection calculations in three-dimensional space.

Additional Insights and Applications

Why is this approach important? Understanding how to derive these equations is fundamental in various applications:

- Engineering Design: Calculating the trajectory of components perpendicular to surfaces.

- Computer Graphics: Rendering objects with precise orientation.
- Physics: Modeling motion along paths orthogonal to fields or surfaces.
- Mathematics Education: Building intuition about the relationships between points, lines, and planes.

Extensions and Variations:

- Finding the intersection point of the line with the plane: Substituting the parametric equations into the plane equation.
- Generalizing to other points and planes: Adjusting the point and normal vector accordingly.
- Analyzing multiple perpendicular lines: For more complex surfaces or constraints.

Conclusion

Mastering the derivation of parametric equations for lines in three dimensions hinges on a clear understanding of the geometric principles involved, especially the relationship between lines and planes. In this detailed exploration, we've established that:

- The key to perpendicularity is aligning the line's direction vector with the plane's normal vector.
- The specific point through which the line passes anchors the parametric equations.
- Assumptions about the plane's form are sometimes necessary when the original equation is ambiguous or complex.

By following the methodical steps outlined—identifying the normal vector, establishing the direction, and writing the parametric equations—you gain a robust framework applicable to a wide range of geometric problems in 3D space. Whether for academic pursuits or practical applications, this approach equips you with the tools to navigate the three-dimensional world with confidence and precision.

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