Find The Arc Length Of The Curve On The Given Interval. (round Your Answer To Three Decimal Places.)

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Understanding how to calculate the arc length of a curve is a fundamental concept in calculus, essential for various applications in engineering, physics, and mathematics. Whether you're analyzing the length of a roller coaster track, the path of a satellite, or the trajectory of a projectile, knowing how to determine the length of a curve over a specific interval is invaluable. This article provides a comprehensive guide to finding the arc length of a curve on a given interval, with step-by-step instructions, formulas, and practical examples, all rounded to three decimal places as required.

Introduction to Arc Length of a Curve

Before diving into the calculation methods, it's important to understand what arc length represents. In simple terms, the arc length of a curve between two points is the distance measured along the curve itself, not just the straight-line distance between those points.

Key Concepts:

- Curve: A function y = f(x) or parametric equations describing a path.
- Interval: The segment [a, b] over which the arc length is calculated.
- Arc Length: The actual length of the curve between x = a and x = b.

Mathematical Foundation of Arc Length Calculation

The general formula for the arc length L of a function $y = f(x)$ over the interval $[a, b]$ is derived from the Pythagorean theorem and calculus principles:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Similarly, for parametric curves defined by $x = x(t)$ and $y = y(t)$ over $t \in [t_1, t_2]$:

$$
L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \, dt
$$

This formula accounts for the infinitesimal segment of the curve, summing these segments over the interval to find the total length.

Step-by-Step Procedure to Find Arc Length

Calculating the arc length involves several systematic steps:

1. Identify the function or parametric equations

- For a function: $y = f(x)$
- For parametric equations: $x = x(t)$, $y = y(t)$

2. Find the derivative(s)

- For $y = f(x)$, compute $\frac{dy}{dx}$.
- For parametric equations, compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

3. Set up the arc length integral

- For single-variable functions:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

- For parametric equations:

$$
L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \, dt
$$

4. Simplify the integrand

- Simplify the expression under the square root for easier integration.

5. Evaluate the integral

- Use analytical methods or numerical techniques (such as Simpson's rule or trapezoidal rule) if the integral does not have an elementary antiderivative.

6. Round the result to three decimal places

- Ensure the final answer is rounded appropriately for precision.

Practical Examples

Let's explore some practical examples to solidify understanding.

Example 1: Find the arc length of $y = x^2$ on [0, 1]

Step 1: Function and interval

- $y = x^2$, $a = 0$, $b = 1$

Step 2: Derivative

$$
\frac{dy}{dx} = 2x
$$

Step 3: Set up the integral

$$
L = \int_{0}^{1} \sqrt{1 + (2x)^2} \, dx = \int_{0}^{1} \sqrt{1 + 4x^2} \, dx
$$

Step 4: Simplify and evaluate

The integral:

$$
L = \int_{0}^{1} \sqrt{1 + 4x^2} \, dx
$$

This is a standard form. Use a substitution:

Let $x = \frac{1}{2} \sinh u$, then $dx = \frac{1}{2} \cosh u \, du$.

When $x=0$, $u=0$. When $x=1$, $u= \sinh^{-1}(2)$.

Express the integrand:

$$
\sqrt{1 + 4x^2} = \sqrt{1 + 4 \left(\frac{1}{2} \sinh u \right)^2 } = \sqrt{1 + \sinh^2 u} = \cosh u
$$

Thus,

$$
L = \int_{u=0}^{\sinh^{-1}(2)} \cosh u \times \frac{1}{2} \cosh u \, du = \frac{1}{2} \int_{0}^{\sinh^{-1}(2)} \cosh^2 u \, du
$$

Using the identity:

$$
\cosh^2 u = \frac{1 + \cosh 2u}{2}
$$

So,

$$
L = \frac{1}{2} \int_{0}^{\sinh^{-1}(2)} \frac{1 + \cosh 2u}{2} \, du = \frac{1}{4} \int_{0}^{\sinh^{-1}(2)} (1 + \cosh 2u) \, du
$$

Integrate:

$$
L = \frac{1}{4} \left[u + \frac{1}{2} \sinh 2u \right]_{0}^{\sinh^{-1}(2)}
$$

Calculate $u = \sinh^{-1}(2)$:

$$
\sinh^{-1}(2) \approx 1.4436
$$

Compute $\sinh 2u$:

$$
\sinh 2u = 2 \sinh u \cosh u
$$

Since $\sinh u = 2$, because $u = \sinh^{-1}(2)$, then:

$$
\cosh u = \sqrt{1 + \sinh^2 u} = \sqrt{1 + 4} = \sqrt{5} \approx 2.2361
$$

Therefore:

$$
\sinh 2u = 2 \times 2 \times 2.2361 = 8.9442
$$

Now, plug in:

$$
L \approx \frac{1}{4} \left[1.4436 + \frac{1}{2} \times 8.9442 \right] = \frac{1}{4} \left[1.4436 + 4.4721 \right] = \frac{1}{4} \times 5.9157 = 1.4789
$$

Final answer (rounded to three decimal places):

$$
\boxed{L \approx 1.479}
$$

Example 2: Find the arc length of the parametric curve $x = t^2$, $y = t^3$ over $t \in [0, 1]$

Step 1: Derivatives:

$$
\frac{dx}{dt} = 2t,\quad \frac{dy}{dt} = 3t^2
$$

Step 2: Set up the integral:

$$
L = \int_{0}^{1} \sqrt{(2t)^2 + (3t^2)^2} \, dt = \int_{0}^{1} \sqrt{4t^2 + 9t^4} \, dt
$$

Factor:

$$
L = \int_{0}^{1} t \sqrt{4 + 9 t^2} \, dt
$$

Step 3: Substitution:

Let $u = 4 + 9 t^2$

Then,

$$
du = 18 t \, dt \Rightarrow t \, dt = \frac{du}{18}
$$

When $t=0$, $u=4$. When $t=1$, $u=4 + 9=13$.

Rewriting the integral:

$$

$$L = \int_{u=4}^{13} \sqrt{u} \times \frac{du}{18}$$

Frequently Asked Questions

How do I find the arc length of a curve y = f(x) on the interval [a, b]?

To find the arc length, use the formula $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$, where $f'(x)$ is the derivative of $f(x)$.

What steps should I follow to compute the arc length of a given curve?

First, find the derivative of the function, then set up the arc length integral with the limits of the interval, evaluate the integral, and round the result to three decimal places.

Can you provide an example of calculating the arc length for $y = x^2$ from $x = 0$ to $x = 1$?

Yes. First, find $f'(x) = 2x$. Then, set up the integral $L = \int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx$. Evaluating this integral and rounding to three decimal places gives approximately 1.478.

Why is it necessary to square the derivative in the arc length formula?

Squaring the derivative accounts for the slope of the curve, ensuring the length calculation includes both horizontal and vertical changes, which is essential for accurately measuring the curve's length.

Are there any tools or software recommended for calculating complex arc length integrals?

Yes, mathematical software like WolframAlpha, Desmos, or graphing calculators can help evaluate complex integrals numerically, providing accurate results rounded to three decimal places.

Additional Resources

Find the Arc Length of the Curve on the Given Interval is a common problem in calculus that combines the concepts of derivatives and integrals to determine the length of a curve between two points. Whether you're studying for an exam, working on a project, or simply seeking a deeper understanding of calculus, mastering how to find the arc length of a curve is an essential skill. This process involves understanding the geometric interpretation of derivatives and applying the arc length formula, which can seem daunting at first but becomes straightforward with a structured approach.

Understanding the Concept of Arc Length

Before diving into the calculation steps, it's important to understand what the arc length represents. Think of it as the actual length of a curved line between two points, similar to measuring the length of a winding road or a roller coaster track. Unlike straight-line distance, which can be found using the Pythagorean theorem, the length of a curve requires integrating the infinitesimal segments along its path.

Why is Arc Length Important?

- In Engineering and Physics: To measure the length of a wire, a cable, or a path taken by an object.
- In Geometry: To understand the properties of curves.
- In Calculus: To apply integral calculus to real-world problems involving curves.

The Fundamental Formula for Arc Length

The core of finding the arc length of a curve $y = f(x)$ over an interval $[a, b]$ is given by the formula:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

where:

- L is the arc length.
- a and b are the bounds of the interval on the x-axis.
- $\frac{dy}{dx}$ is the derivative of the function $f(x)$.

This formula arises from considering an infinitesimal segment of the curve as a right triangle with sides dx and dy, then summing these tiny segments across the interval.

Step-by-Step Guide to Find the Arc Length

1. Identify the Function and Interval

Begin by clearly defining the function $y = f(x)$ and the interval $[a, b]$ over which you need to find the arc length.

Example:
Suppose $y = x^2$, and the interval is $[0, 2]$.

2. Compute the Derivative dy/dx

Differentiate the function with respect to x.

Example:

$$
\frac{dy}{dx} = 2x
$$

3. Set Up the Arc Length Integral

Using the formula:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Substitute the derivative into the integrand.

Example:

$$
L = \int_{0}^{2} \sqrt{1 + (2x)^2} \, dx = \int_{0}^{2} \sqrt{1 + 4x^2} \, dx
$$

4. Simplify the Integrand

Identify if the integral resembles a standard form or if substitution is needed. For the example:

$$
\sqrt{1 + 4x^2}
$$

This resembles a form suitable for a hyperbolic substitution or trigonometric substitution.

5. Choose an Appropriate Substitution

Common substitutions include:

- Trigonometric substitution: For integrals involving $\sqrt{a^2 + x^2}$, $x = a \tan \theta$.
- Hyperbolic substitution: For $\sqrt{x^2 + a^2}$, $x = a \sinh t$.

Example:
For $\sqrt{1 + 4x^2}$, set:

$$
x = \frac{1}{2} \tan \theta \Rightarrow dx = \frac{1}{2} \sec^2 \theta \, d\theta
$$

Transform the integral accordingly and simplify.

6. Perform the Integration

Carry out the substitution, simplify the integral, and compute it. Be attentive to:

- Changing the limits of integration if you substitute x with θ.
- Simplifying the radical expressions.

- Recognizing standard integral forms or using integral tables.

Example:
Using $x = \frac{1}{2} \sinh t$:

$$
\sqrt{1 + 4x^2} = \sqrt{1 + 4 \times \frac{1}{4} \sinh^2 t} = \sqrt{1 + \sinh^2 t} = \cosh t
$$

And $dx = \frac{1}{2} \cosh t\, dt$.

The integral becomes:

$$
L = \int \cosh t \times \frac{1}{2} \cosh t\, dt = \frac{1}{2} \int \cosh^2 t\, dt
$$

Use identities to integrate $\cosh^2 t$.

7. Back-Substitute to the Original Variable

After integration, substitute back to x to evaluate the definite integral over the original bounds.

8. Round Your Final Answer

Once you obtain the exact value of the integral, round your answer to three decimal places as required.

Practical Tips for Calculating Arc Length

- Identify the integrand: Recognize standard forms or use substitution techniques.
- Use symmetry: If the function is symmetric, leverage that to simplify calculations.
- Check the derivative: Sometimes simplifying $\left(\frac{dy}{dx} \right)^2$ makes the integral easier.
- Use a calculator or software: For complex integrals, tools like WolframAlpha, Desmos, or graphing calculators can assist.
- Verify your bounds: Remember to adjust the limits after substitution if you change variables.

Example Problem: Find the Arc Length of $y = \sqrt{x}$ on $[0, 4]$

Step 1: Function and interval:

$$
y = \sqrt{x}, \quad a=0, \quad b=4
$$

Step 2: Derivative:

$$
\frac{dy}{dx} = \frac{1}{2\sqrt{x}}
$$

Step 3: Set up the integral:

$$
L = \int_{0}^{4} \sqrt{1 + \left(\frac{1}{2\sqrt{x}} \right)^2} \, dx = \int_{0}^{4} \sqrt{1 + \frac{1}{4x}} \, dx
$$

Step 4: Simplify integrand:

$$
\sqrt{ \frac{4x + 1}{4x} } = \frac{\sqrt{4x + 1}}{2 \sqrt{x}}
$$

Step 5: Rewrite the integral:

$$
L = \int_{0}^{4} \frac{\sqrt{4x + 1}}{2 \sqrt{x}} \, dx
$$

Step 6: Substitution:

Let $t = \sqrt{x} \Rightarrow x = t^2, dx = 2t\, dt$

Change bounds:

- When $x=0$, $t=0$
- When $x=4$, $t=2$

Express numerator:

$$
\sqrt{4t^2 + 1}
$$

The integral becomes:

$$
L = \int_{0}^{2} \frac{\sqrt{4t^2 + 1}}{2t} \times 2t \, dt = \int_{0}^{2} \sqrt{4t^2 + 1} \, dt
$$

Step 7: Simplify and integrate:

$$
L = \int_{0}^{2} \sqrt{4t^2 + 1} \, dt
$$

Use substitution:

$$
u = 2t \Rightarrow du=2 dt \Rightarrow dt= \frac{du}{2}
$$

Bounds:

- $t=0 \Rightarrow u=0$
- $t=2 \Rightarrow u=4$

Express the integral:

$$
L = \int_{u=0}^{4} \sqrt{u^2 + 1} \times \frac{du}{2} = \frac{1}{2} \int_{0}^{4} \sqrt{u^2 + 1} \, du
$$

The integral:

$$
\int \sqrt{u^2 + 1} \, du
$$

is standard, with solution:

$$
\frac{u}{2} \sqrt{u^2 + 1} + \frac{1}{2} \sinh^{-1} u + C
$$

Step 8: Evaluate the integral:

$$
L = \frac{1}{2} \left[\frac{u}{2} \sqrt{u^2 + 1} + \frac{1}{2} \sinh^{-1} u \right]_0^4
$$

Calculate at $u=4$:

\[

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