Find The Average Value Of F(x, Y, Z) = Z Over The Region Bounded Below By The Xy-plane, On The Sides

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Understanding how to find the average value of a multivariable function over a specific region is fundamental in multivariable calculus. In this guide, we will explore the process of calculating the average value of the function $F(x, y, z) = z$ over a three-dimensional region bounded below by the xy-plane and on the sides by other surfaces or boundaries. This process involves a combination of volume integrals and understanding the geometric boundaries of the region. Whether you're a student preparing for calculus exams or a professional applying these concepts in engineering or physics, this comprehensive overview will clarify the steps involved.

Understanding the Problem Setup

Before diving into calculations, it’s essential to comprehend the problem's geometric and mathematical context.

What is the Average Value of a Function?

The average value of a function $F(x, y, z)$ over a region R in three-dimensional space is given by:

$$
\text{Average value} = \frac{1}{\text{Volume of R}} \iiint_R F(x, y, z) \, dV
$$

This formula indicates that to find the average, you need to:
- Compute the volume of the region R.
- Calculate the triple integral of the function over R.
- Divide the integral by the volume.

Region R: Bounded Below by the XY-plane

The region R is defined with the following properties:
- The bottom boundary is the xy-plane, i.e., $z = 0$.
- The top boundary and sides are determined by other surfaces or constraints, which define the shape of R.

Understanding the boundaries is crucial because it influences the limits of integration. Typically, the region might be bounded above by a surface $z = f(x, y)$, and on the sides by curves or planes.

Step-by-Step Approach to Find the Average Value

Calculating the average value involves several systematic steps:

1. Identify the Region R

- Determine the boundary surfaces.
- Express the region in terms of inequalities involving x, y, and z.

2. Set Up the Volume Integral

- Choose an appropriate order of integration (dz dy dx, dx dz dy, etc.).
- Write the limits of integration based on the region's boundaries.

3. Compute the Volume of R

- Integrate the differential volume element dV over R to find volume.

4. Evaluate the Integral of F(x, y, z) = z over R

- Set up the triple integral of z over the volume R.
- Integrate in the chosen order.

5. Calculate the Average Value

- Divide the integral of the function by the volume obtained earlier.

Practical Example: A Step-by-Step Calculation

Let's consider a concrete example for clarity:

Suppose the region R is bounded below by the xy-plane ($z=0$), and on the sides by the paraboloid $z = x^2 + y^2$, with x and y bounded such that the paraboloid intersects the plane $z=1$.

Step 1: Describe the Region R

- The paraboloid $z = x^2 + y^2$.

- The upper boundary $z = 1$.
- Since the paraboloid intersects the plane $z=1$ when $x^2 + y^2 = 1$, the projection of R onto the xy-plane is the disk $x^2 + y^2 \leq 1$.

Step 2: Set Up the Limits of Integration

- Convert to cylindrical coordinates for simplicity:

$$
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
$$

- The bounds are:

- θ from 0 to 2π,
- r from 0 to 1,
- z from 0 to r^2 (since the paraboloid is $z = r^2$) for the lower boundary, and to 1 for the upper boundary.

However, because the volume is between the xy-plane and the paraboloid, and capped at z=1:

- For each point (r, θ), z varies from 0 to r^2 (paraboloid), but only up to z=1. Since $r^2 \leq 1$, the maximum z on the paraboloid is at r=1, z=1.

Step 3: Express the Volume Element

- In cylindrical coordinates, $dV = r \, dr \, d\theta \, dz$.

Step 4: Compute the Volume of R

Since the volume is bounded above by the paraboloid $z = r^2$ and below by the xy-plane $z=0$, the volume integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{r^2} r \, dz \, dr \, d\theta
$$

Calculate the inner integral:

$$
\int_{0}^{r^2} r \, dz = r \left[z\right]_0^{r^2} = r \cdot r^2 = r^3
$$

Now the volume integral:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} r^3 \, dr \, d\theta
$$

Integrate with respect to r:

$$
\int_{0}^{1} r^3 \, dr = \frac{r^4}{4} \bigg|_0^1 = \frac{1}{4}
$$

And with respect to θ:

$$
V = \int_{0}^{2\pi} \frac{1}{4} \, d\theta = \frac{1}{4} \times 2\pi = \frac{\pi}{2}
$$

Thus, the volume of R is $\frac{\pi}{2}$.

Step 5: Compute the Triple Integral of F(x, y, z) = z

In cylindrical coordinates:

$$
\iiint_R z \, dV = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{r^2} z \cdot r \, dz \, dr \, d\theta
$$

Inner integral:

$$
\int_{0}^{r^2} z \cdot r \, dz = r \int_{0}^{r^2} z \, dz = r \left[\frac{z^2}{2}\right]_0^{r^2} = r \cdot \frac{(r^2)^2}{2} = r \cdot \frac{r^4}{2} = \frac{r^5}{2}
$$

Now, integrate over r:

$$
\int_0^{1} \frac{r^5}{2} \, dr = \frac{1}{2} \left[\frac{r^6}{6}\right]_0^1 = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}
$$

Finally, over θ:

$$
\int_0^{2\pi} \frac{1}{12} \, d\theta = \frac{1}{12} \times 2\pi = \frac{\pi}{6}
$$

The integral of z over R is $\frac{\pi}{6}$.

Step 6: Calculate the Average Value

$$
\text{Average} = \frac{\text{Integral of } z}{\text{Volume}} = \frac{\pi/6}{\pi/2} = \frac{\pi/6}{\pi/2} = \frac{\pi/6 \times 2}{\pi} = \frac{2}{6} = \frac{1}{3}
$$

Therefore, the average value of $F(x,y,z) = z$ over this region is $\frac{1}{3}$.

Additional Tips for Solving Similar Problems

To efficiently compute the average value of a function over a complex region, consider the following:

1. Visualize the Region

- Sketch the boundaries and the region if possible.
- Understand the geometric shape to choose the best coordinate system.

2. Choose the Appropriate Coordinate System

- Cartesian coordinates work well for rectangular bounds.
- Cylindrical coordinates are ideal for circular or cylindrical symmetry.
- Spherical coordinates suit spherical regions.

3. Determine Limits Carefully

- Express the bounds explicitly based on the surfaces.
- Always verify the inequalities.

4. Simplify the Integral

- Use symmetry where possible.
- Factor constants outside the integral.

5. Confirm the Region's Volume

- Computing the volume helps interpret the average value.
- Double-check limits

Frequently Asked Questions

What is the formula to find the average value of the function F(x, y, z) = z over a region in three dimensions?

The average value of F over a region R is given by (1/Volume of R) times the triple integral of F over R, i.e., Average = $(1/\iiint_R dV) \iiint_R F(x,y,z)\, dV$.

How do you determine the volume of the region bounded below by the xy-plane and on the sides in a three-dimensional setting?

The volume is obtained by integrating the area of the cross-sectional regions over the height, typically using triple integrals with appropriate bounds based on the region's shape.

In the context of finding the average value of F(x, y, z) = z, what role does the projection onto the xy-plane play?

The projection helps define the base region over which the volume and the integral of z are calculated, simplifying the setup of the triple integral.

What are the typical steps to compute the average value of F(x, y, z) = z over a bounded region?

First, define the region and set up the triple integral of z over that region. Then, compute the volume by integrating 1 over the region. Finally, divide the triple integral of z by the volume to find the average.

How does the boundary condition 'bounded below by the xy-plane' influence the limits of integration?

It sets the lower limit of z as zero, ensuring that the integration for z starts from 0 up to the upper boundary surface, which defines the region's height.

Can symmetry be used to simplify the calculation of the average value of z in certain regions?

Yes, if the region is symmetric about the xy-plane or other axes, symmetry can be used to simplify integrals or determine that certain parts cancel out, reducing calculation complexity.

What considerations are important when choosing coordinate systems for calculating the average value in such regions?

Choosing coordinate systems like cylindrical or spherical coordinates may simplify the bounds and integrals, especially if the region has symmetric or circular features.

How does the shape of the boundary surfaces influence the setup of the triple integral for finding the average of

F(x, y, z) = z?

The shape determines the limits of integration for each variable. Accurately describing the boundary surfaces ensures correct limits, which are crucial for precise calculation of the integral and average.

What is the significance of the 'region bounded on the sides' in the context of calculating the average value?

It defines the lateral boundaries of the region, which determine the cross-sectional area and influence the limits of integration for x and y, ultimately impacting the calculation of the average value.

Additional Resources

Find The Average Value Of F(x, y, z) = z Over The Region Bounded Below By The XY-Plane, On The Sides

Understanding how to find the average value of a function over a specified region is a fundamental concept in multivariable calculus. Specifically, for the function $F(x, y, z) = z$, calculating its average value over a region bounded below by the xy-plane and on the sides involves a combination of volume integration and the principles of averaging functions in three dimensions. This problem offers valuable insights into volume integration, triple integrals, and the geometric interpretation of averages in multivariable contexts.

Introduction to the Problem

The task is to find the average value of the function $F(x, y, z) = z$ over a certain three-dimensional region. The region is specified as being "bounded below by the xy-plane," which indicates that the lower boundary of this volume is the plane $z=0$. The phrase "on the sides" suggests the region may be bounded by some vertical surfaces, possibly involving specific curves or planes that form the sides of the region.

Key points:

- The region of integration is three-dimensional.
- The function $F(x,y,z) = z$ is a simple linear function in the z-direction.
- The bounds are defined by the xy-plane and some side surfaces.

Understanding the nature of the bounding surfaces is crucial. Typically, such problems involve a region that might be a solid bounded by surfaces like cylinders, cones, or planes, and the goal is to compute the average value of z over this volume.

Defining the Region of Integration

Geometric Description

Before computing the average value, it is essential to understand the shape and boundaries of the region:

- Lower boundary: The xy-plane, described mathematically as $z=0$.
- Side boundaries: These could involve various surfaces such as vertical planes, cylindrical surfaces, or conical surfaces.

Suppose, for example, the region is a solid bounded by:

- The xy-plane ($z=0$)
- A cylindrical surface $r = a$ (if in cylindrical coordinates)
- A plane inclined or vertical, such as $z = f(x, y)$, defining the "sides."

In many problems, the region might be a cone or a cylinder segment, which simplifies the integration process.

Note: Since the problem statement is abstract, the typical approach is to assume a specific region or define the general method applicable to various regions.

Mathematical Description of the Region

Suppose the region B is described as:

$$
B = \left\{(x, y, z) \mid (x, y) \in D,\ 0 \leq z \leq g(x, y) \right\}
$$

where:

- D is a bounded region in the xy-plane.
- $g(x, y)$ describes the upper boundary surface (the "side" boundary in the vertical direction).

For example, D could be a circle $x^2 + y^2 \leq R^2$, and $g(x, y)$ could be a function defining the height of the region over D.

Calculating the Volume of the Region

The first step in finding the average value of $F(x, y, z) = z$ over the region is to compute the volume V of the region B:

$$
V = \iiint_B dV
$$

Depending on the bounds, this integral can be set up in different coordinate systems:

- Cartesian coordinates: x, y, z
- Cylindrical coordinates: (r, θ, z)

Volume Integral in Cartesian Coordinates

If the bounds are explicitly given in Cartesian coordinates, the volume integral takes the form:

$$
V = \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} \int_{0}^{g(x,y)} dz\, dy\, dx
$$

Since the integrand for volume is $dV = dz\, dy\, dx$, integrating dz over $[0, g(x,y)]$ yields:

$$
V = \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} g(x,y) \, dy \, dx
$$

Calculating the Average Value of $F(x, y, z) = z$

The average value F_{avg} of a function F over a volume V is given by:

$$
F_{avg} = \frac{1}{V} \iiint_B F(x, y, z) \, dV
$$

In our case:

$$
F_{avg} = \frac{1}{V} \iiint_B z \, dV
$$

This involves two integrals: one to compute the total volume V, and the other to compute the integral of z over the volume.

Integral of z over the Region

Expressed explicitly:

$$
\iiint_B z \, dV = \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} \int_{0}^{g(x,y)} z \, dz \, dy \, dx
$$

Integrating with respect to z:

$$
\int_0^{g(x,y)} z \, dz = \frac{1}{2} g(x,y)^2
$$

Thus,

$$
\iiint_B z \, dV = \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} \frac{1}{2} g(x,y)^2 \, dy\, dx
$$

Putting It All Together: Calculating the Average

The average value becomes:

$$
F_{avg} = \frac{\displaystyle \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} \frac{1}{2} g(x,y)^2 \, dy\, dx}{\displaystyle \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} g(x,y) \, dy\, dx}
$$

Key steps:

1. Determine the bounds x_{min}, x_{max}, $y_{min}(x)$, $y_{max}(x)$.
2. Express $g(x,y)$, the height function of the boundary.
3. Perform the double integrals to compute numerator and denominator.
4. Divide the two results to find the average.

Coordinate System Choice and Simplification

Depending on the symmetry of the region, choosing an appropriate coordinate system simplifies the integration:

- Cylindrical coordinates are useful if the region is circular or radial in nature, i.e., $x^2 + y^2 \leq R^2$.
- Polar coordinates in 2D for the xy-plane:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

with $r \in [0, R]$, $\theta \in [0, 2\pi]$.

In cylindrical coordinates, the volume element becomes:

$$
dV = r \, dr \, d\theta \, dz
$$

and the integrals adapt accordingly.

Example Calculation: A Specific Region

Suppose the region B is a right circular cone with vertex at the origin, extending up to height h, bounded below by the xy-plane and sides by the cone surface:

$$
z = k \sqrt{x^2 + y^2}
$$

for some constant k, with $0 \leq z \leq h$.

In cylindrical coordinates:

$$
z = k r
$$

and the boundary occurs at $z = h$, i.e., $r = h / k$.

The region in cylindrical coordinates:

$$
0 \leq r \leq \frac{h}{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq k r
$$

\]

The volume:

\[
V = \int_{0}^{2\pi} \int_{0}^{h/k} \int_{0}^{k r} r \, dz\, dr\, d\theta
\]

Integrate with respect to \(z\):

\[
\int_{0}^{k r} r \, dz = r \times (k r) = k r^2
\]

Complete the volume integral:

\[
V = \int_{0}^{2\pi} \int_{0}^{h/k} k r^2 \, dr \, d\theta = 2\pi \int_{0}^{h/k

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