

Find The Bases For Col A And Nul A, And Then State The Dimension Of These Subspaces For The Matrix A

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When studying linear algebra, understanding the structure of a matrix involves exploring its column space and null space. These subspaces provide insight into the solutions of linear systems and the properties of the matrix itself. Specifically, identifying bases for the column space (Col A) and the null space (Nul A) allows us to characterize these subspaces explicitly. Moreover, determining their dimensions—the rank and nullity—gives us vital information about the matrix's rank, the number of independent columns, and the number of solutions to associated linear systems.

In this comprehensive article, we will explore how to find the bases for Col A and Nul A for a given matrix A, how to compute their dimensions, and the significance of these concepts in linear algebra.

Understanding the Column Space and Null Space

Before diving into the methods for finding bases, it's essential to understand what the column space and null space represent.

Column Space (Col A)

The column space of a matrix A, denoted as Col A, is the subspace of $\mathbb{R}^m$ (for an $m \times n$ matrix A) spanned by the columns of A. Essentially, it consists of all linear combinations of the columns of A:

$$\text{Col } A = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$$

where $\mathbf{a}_i$ are the column vectors of A.

Null Space (Nul A)

The null space of A, denoted as Nul A, is the set of all solutions $\mathbf{x}$ to the homogeneous equation:

$$\{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$$

This subspace of $\mathbb{R}^n$ contains all vectors $\mathbf{x}$ such that the linear transformation represented by A maps $\mathbf{x}$ to the zero vector.

Steps to Find the Basis for Col A

Finding a basis for the column space involves identifying a set of linearly independent columns that span the entire column space.

1. Write the matrix A

Start with the matrix whose column space you want to analyze.

2. Perform row operations to reduce A to its echelon form

Use Gaussian elimination (or Gauss-Jordan elimination) to convert A into a row echelon form (REF) or reduced row echelon form (RREF). This simplifies the identification of pivot columns.

3. Identify pivot columns

In the RREF matrix, each leading 1 (pivot) indicates a pivot column. The original columns corresponding to these pivot columns form a basis for the column space.

4. Extract the basis vectors

Select the original columns of A that correspond to the pivot columns in the RREF. These columns are linearly independent and span the column space.

Example: Finding the Basis for Col A

Suppose

```

\l
A = \begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 4 \\
0 & 0 & 0
\end{bmatrix}
\l

```

Perform row operations, but since A is already in echelon form, the pivot columns are 1 and 2 (the first and second columns).

- The pivot columns in the original matrix are columns 1 and 2:

```

\l
\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad
\mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}
\l

```

- Therefore, the basis for Col A is $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Steps to Find the Basis for Nul A

To find a basis for the null space, follow these steps:

1. Write the augmented matrix for the homogeneous system

Since the system is $A\mathbf{x} = \mathbf{0}$, set up the augmented matrix $[A \mid \mathbf{0}]$.

2. Reduce A to RREF

Perform Gaussian elimination to obtain the reduced form, which makes free variables explicit.

3. Identify free variables and leading variables

- Pivot columns correspond to leading variables.
- Non-pivot columns correspond to free variables.

4. Express basic variables in terms of free variables

Write the solutions to the system in parametric form, assigning parameters to free variables.

5. Write the solution vectors as linear combinations of free variables

Each free variable corresponds to a basis vector for $\text{Nul } A$.

Example: Finding a Basis for $\text{Nul } A$

Using the matrix from above:

```
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 4 \\
0 & 0 & 0
\end{bmatrix}
\]
```

The homogeneous system:

```
\[
\begin{cases}
x_1 + 2x_2 + 3x_3 = 0 \\
x_2 + 4x_3 = 0
\end{cases}
\]
```

- From the second equation:

```
\[
x_2 = -4x_3
\]
```

- Substitute into the first:

```
\[
x_1 + 2(-4x_3) + 3x_3 = 0 \Rightarrow x_1 - 8x_3 + 3x_3 = 0 \Rightarrow x_1 = 5x_3
\]
```

- Let $(x_3 = t)$ (free parameter). Then:

```
\[
```

$$x_1 = 5t, \quad x_2 = -4t, \quad x_3 = t$$

- The general solution vector is:

$$\mathbf{x} = t \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

- The basis for Nul A is $\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$.

Determining the Dimensions of the Column Space and Null Space

The dimensions of these subspaces are fundamental in understanding the matrix's properties.

1. Rank of A (dim of Col A)

The rank of A is the number of pivot columns in its echelon form. It equals the maximum number of linearly independent columns.

- Dimension of Col A = Rank(A)

2. Nullity of A (dim of Nul A)

The nullity is the number of free variables in the solution of $(A\mathbf{x} = \mathbf{0})$, which corresponds to the number of non-pivot columns.

- Nullity(A) = n - Rank(A)

where (n) is the number of columns of A.

3. Rank-Nullity Theorem

This fundamental theorem states:

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

This relation helps verify the correctness of calculations.

Summary and Significance

Understanding how to find the bases for the column space and null space of a matrix A is central to linear algebra. These bases not only provide explicit descriptions of the subspaces but also facilitate computations related to solving linear systems, understanding the invertibility of matrices, and analyzing linear transformations.

- Bases for $\text{Col } A$ are obtained by selecting the original columns corresponding to pivot columns after row reduction. These columns are linearly independent and span the column space.
- Bases for $\text{Nul } A$ are derived from the parametric solutions to the homogeneous system, with each basis vector corresponding to a free variable.

The dimensions of these subspaces—the rank and nullity—are key parameters that describe the matrix's properties. The rank indicates the maximum number of linearly independent columns, and the nullity measures the degree of freedom in solutions to the homogeneous equation.

Practical Applications

The concepts discussed have numerous applications:

- Solving Linear Systems: Knowing the bases helps in understanding the solution space structure.
- Data Analysis: Dimensionality reduction techniques often rely on bases of subspaces.
- Computer Graphics: Transformation properties are analyzed via column and null spaces.
- Engineering: Signal processing and control systems utilize these concepts for system analysis.

By mastering the procedures to find these bases and their dimensions, students and professionals can analyze matrices more effectively and apply linear algebra techniques confidently.

Conclusion

Finding the bases for the column space and null space of a matrix A is a fundamental skill in linear algebra. It involves systematic procedures—row reduction, identifying pivot columns, solving homogeneous systems—and leads to insights about the matrix's structure. The dimensions, given by the rank and nullity, are crucial in understanding the solutions to systems of equations and the properties of the linear transformations represented by A .

Whether

Frequently Asked Questions

How do I find the basis for Col A (column space) of a matrix A ?

To find the basis for Col A , identify all the pivot columns in the matrix's row echelon form. The original columns corresponding to these pivot columns form the basis for the column space.

What is the method to determine the basis for Nul A (null space) of a matrix A ?

To find the basis for Nul A , solve the homogeneous system $Ax = 0$, express the solution in parametric form, and the set of free variables' vectors forms the basis for the null space.

How do I compute the dimension of the column space (rank) of matrix A ?

The dimension of the column space, known as the rank, equals the number of pivot columns in the row echelon form of A .

What is the relationship between the dimensions of Col A and Nul A for matrix A ?

According to the Rank-Nullity Theorem, the sum of the dimension of Col A (rank) and the dimension of Nul A (nullity) equals the number of columns of A .

Can you illustrate how to find a basis for both Col A and Nul A with an example?

Yes, for example, given A , first row reduce to echelon form to find pivot columns for Col A basis, and solve $Ax=0$ for Nul A basis by expressing free variables. The basis vectors are then extracted from the solutions.

Why is it important to find the basis and dimension of

these subspaces?

Determining the basis and dimension helps understand the structure of the matrix, such as its rank, nullity, and the solutions to associated linear systems, which are fundamental in linear algebra applications.

What tools or techniques are most effective for finding bases for Col A and Nul A?

Row reduction to echelon form, identifying pivot columns, and solving homogeneous systems are the most effective techniques for finding these bases.

Is the basis for Nul A always composed of vectors in parametric form?

Yes, the basis vectors for Nul A are typically expressed in parametric form derived from the free variables in the homogeneous system solution.

Additional Resources

Find The Bases For Col A And Nul A, And Then State The Dimension Of These Subspaces For The Matrix A

Understanding the structure of a matrix is fundamental in linear algebra, especially when analyzing the properties of linear transformations. Two key subspaces associated with any matrix are the column space (Col A) and the null space (Nul A). Identifying their bases and dimensions offers deep insights into the matrix's behavior, rank, and nullity, which are essential for solving systems of equations, understanding invertibility, and more. This article provides a comprehensive yet accessible guide to finding the bases for these subspaces and understanding their dimensions, illustrating the process with detailed explanations and practical steps.

The Importance of Col A and Nul A in Linear Algebra

Before diving into the methods of finding bases, it's crucial to understand what these subspaces represent:

- Column Space (Col A): This is the set of all possible linear combinations of the columns of matrix A. Geometrically, it represents the range of the linear transformation defined by A. The dimension of Col A, known as the rank of A, indicates how many dimensions the output space of the transformation occupies.

- Null Space (Nul A): Also called the kernel, this is the set of all solutions to the homogeneous system $Ax = 0$. It captures the vectors that are mapped to the zero vector, revealing the degrees of freedom in solutions and the linear dependencies among columns.

Determining bases for these subspaces helps in understanding the structure of solutions, the invertibility of the matrix, and the relationships among columns.

Step 1: Understanding the Matrix and Its Structure

Suppose you are given a specific matrix A :

```
\[A = \begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn}
\end{bmatrix}
```

The goal is to find:

1. A basis for Col A
2. A basis for Nul A
3. The dimensions of these subspaces

The process involves a combination of row operations, identifying pivotal columns, and solving homogeneous systems.

Step 2: Finding the Basis for the Column Space (Col A)

2.1. Reduce the Matrix to Row Echelon Form

Begin with Gaussian elimination:

- Use elementary row operations to convert A into a row echelon form (REF) or reduced row echelon form (RREF).
- The purpose is to identify the pivot columns—columns that contain leading entries after elimination.

2.2. Identify Pivot Columns

- The pivot columns correspond to the linearly independent columns of the original matrix.
- These columns form a basis for the column space.

2.3. Extract the Basis Vectors

- The basis for Col A consists of the original columns of A that correspond to the pivot columns identified in the row echelon form.
- These columns are linearly independent and span the entire column space.

2.4. Practical Example

Suppose after row reduction, columns 1, 3, and 4 are pivot columns. Then, the basis for Col A is simply the original columns 1, 3, and 4 of A.

Step 3: Finding the Basis for the Null Space (Nul A)

3.1. Set Up the Homogeneous System

- Write the augmented matrix $[A \mid 0]$, or directly work with A to solve $Ax = 0$.
- Use the row echelon form obtained earlier.

3.2. Express Variables in Terms of Free Variables

- Identify pivot variables (leading variables) and free variables (non-pivot columns).
- Assign parameters to free variables.

3.3. Write General Solution

- Express each pivot variable in terms of free variables.
- The solutions form the null space, which is a subspace of $\mathbb{R}^n$.

3.4. Find the Basis Vectors

- The null space is spanned by vectors corresponding to each free variable.
- For each free variable, set it to 1 while others are 0, and solve for the pivot variables.
- These resulting vectors form a basis for Nul A.

3.5. Practical Example

Suppose columns 2 and 5 are free variables. By setting each to 1 in turn and solving for the pivot variables, you generate basis vectors for Nul A.

Step 4: Computing Dimensions of the Subspaces

4.1. Rank and Nullity

- The dimension of the column space (rank) equals the number of pivot columns.
- The dimension of the null space (nullity) equals the number of free variables, which is the total number of columns minus the rank.

4.2. Applying the Rank-Nullity Theorem

The Rank-Nullity Theorem states:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where:

- $\text{rank}(A) = \text{dimension of Col } A$
- $\text{rank}(A) + \text{nullity}(A) = \text{dimension of Nul } A$

This relationship helps verify calculations and provides quick dimension insights.

Practical Example: Putting It All Together

Let's consider a concrete example:

Suppose matrix A is:

```
\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
```

Step 1: Row Reduce

The matrix is already in echelon form. Pivot columns are 1 and 2.

Step 2: Find Basis for Col A

- Pivot columns are columns 1 and 2.
- Therefore, basis for Col A is:

```
\[
\left\{
\begin{bmatrix}
1 \\ 0 \\ 0
\end{bmatrix},
\begin{bmatrix}
2 \\ 1 \\ 0
\end{bmatrix}
\right\}
\]
```

or, directly, the first two columns of A:

```
\[
\left\{
\begin{bmatrix}
1 \\ 0 \\ 0
\end{bmatrix},
\begin{bmatrix}
2 \\ 1 \\ 0
\end{bmatrix}
\right\}
\]
```

$\end{bmatrix}$
 $\right\}$
 $\}$

Step 3: Find Basis for Nul A

- Variables: (x_1, x_2, x_3, x_4)
- From the equations:

1. $(x_1 + 2x_2 + 3x_3 + 4x_4 = 0)$
2. $(x_2 + x_4 = 0)$

Express $(x_2 = -x_4)$, and substitute into the first:

$$\begin{aligned}
 & \left[\begin{aligned} x_1 + 2(-x_4) + 3x_3 + 4x_4 &= 0 \end{aligned} \right. \Rightarrow x_1 + (-2x_4) + 3x_3 + 4x_4 = 0 \\
 & \left[\begin{aligned} x_1 + (2x_4) + 3x_3 &= 0 \end{aligned} \right. \Rightarrow x_1 = -2x_4 - 3x_3 \\
 & \left. \right]
 \end{aligned}$$

- Free variables: (x_3, x_4)

Set $(x_3 = s)$, $(x_4 = t)$:

$$\left[\begin{aligned} x_1 &= -2t - 3s, \quad x_2 = -t, \quad x_3 = s, \quad x_4 = t \end{aligned} \right.$$

- Null space basis vectors are obtained by setting (s) and (t) to 1 and 0 in turn:

For $(s=1, t=0)$:

$$\left[\begin{aligned} x &= \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned} \right.$$

For $(s=0, t=1)$:

$$\left[\begin{aligned} x &= \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \right.$$

- Basis for Nul A:

$$\left\{$$

```

\begin{bmatrix}
-3 \ 0 \ 1 \ 0 \\
\end{bmatrix},
\begin{bmatrix}
-2 \ -1 \ 0 \ 1 \\
\end{bmatrix}
\right\}
\}

```

Step 4: Dimensions

- $\dim(\text{Col } A) = 2$
- $\dim(\text{Nul } A) = 2$

since total columns $(n=4)$, and rank $(=2)$, nullity $(=4-2=2)$.

Final Thoughts: The Significance of These Subspaces

Finding the bases and dimensions of Col A and Nul A is more than an academic exercise; it's central to understanding the behavior of linear systems and transformations. The basis vectors serve as building blocks, revealing the fundamental structure, while the dimensions quantify the degrees of freedom and the span of the transformation.

In practical applications—such as computer graphics, engineering

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