Find The Equation Of The Plane In The Form Z = C + Mx + Ny Containing The Line In The Xy-plane Where

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Introduction

In the realm of three-dimensional coordinate geometry, understanding how to determine the equation of a plane that contains a specific line is a fundamental skill. The problem often presents itself in various forms, such as finding the plane that contains a given line and passes through certain points or is parallel to specific vectors. One common and practical form of a plane's equation is:

$$Z = C + Mx + Ny$$

This form is particularly useful because it explicitly relates the z-coordinate to the x and y coordinates, making it straightforward to analyze and visualize the plane in three-dimensional space.

When the line in question lies within the xy-plane, the problem simplifies in some aspects but also introduces unique considerations. The xy-plane itself is described by the equation $z=0$, and any line contained within it must satisfy this condition. The challenge then becomes to find all planes of the form $z = C + Mx + Ny$ that contain this line, which involves understanding the geometric and algebraic relationships between the line and the plane.

This article aims to provide a comprehensive guide to solving such problems, including the step-by-step process, key concepts, and practical examples. Whether you're a student preparing for exams or a professional dealing with three-dimensional modeling, mastering this topic will significantly enhance your understanding of spatial geometry.

Understanding the Basics: Plane Equations and the XY-plane

Standard Equation of a Plane

The general equation of a plane in three-dimensional space can be written as:

$$ Ax + By + Cz + D = 0 $$

where A, B, C are the coefficients determining the orientation of the plane, and D is the constant representing the plane's position relative to the origin.

However, the form we focus on in this article is:

$$ z = C + Mx + Ny $$

This is a special case of the plane equation, often called the intercept form, where:

- C is the z-intercept when $x=0, y=0$,
- M and N are the slopes of the plane in the x and y directions, respectively.

This form is particularly convenient for problems involving the xy-plane because it explicitly shows how z varies with x and y.

The XY-plane and Lines Within It

The xy-plane is characterized by the equation:

$$ z = 0 $$

Any line lying entirely within the xy-plane must satisfy this condition for all points on the line.

Suppose the line in the xy-plane is given parametrically as:

$$
\begin{cases}
x = x_0 + at \\
y = y_0 + bt \\
z = 0
\end{cases}
$$

where (x_0, y_0) is a point on the line, (a, b) is the direction vector, and t is a parameter.

Understanding the line's parametric form is crucial to determining the plane's equation that contains it.

Formulating the Problem: What Are We Looking For?

Given:

- A line L lying entirely in the xy-plane, parametrized as above.
- The requirement to find the plane in the form $z = C + Mx + Ny$ that contains this line.

Our goal is to find the values of C, M, N such that:

1. The plane contains the entire line L.
2. The plane is expressed explicitly as $z = C + Mx + Ny$.

This involves:

- Ensuring that all points on the line satisfy the plane equation.
- Determining the parameters C, M, N based on the line's properties.

Step-by-Step Process to Find the Plane Equation

1. Identify the Line in the XY-plane

Suppose the line L is given parametrically as:

$$
\begin{cases}
x = x_0 + a t \\
y = y_0 + b t \\
z = 0
\end{cases}
$$

where:

- (x_0, y_0) is a point on the line,
- (a, b) is the direction vector.

Alternatively, the line could be expressed in symmetric form:

$$
\frac{x - x_0}{a} = \frac{y - y_0}{b}
$$

with $z=0$.

Key point: All points on the line satisfy $z=0$.

---

2. Write the General Equation of the Plane

Our plane's equation is:

$$ z = C + Mx + Ny $$

Any point (x, y, z) on the plane must satisfy this equation.

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3. Conditions for the Line to Lie in the Plane

Since the entire line lies in the plane, every point $(x, y, 0)$ on the line must satisfy:

$$ 0 = C + Mx + Ny $$

Substituting the parametric form:

$$
0 = C + M(x_0 + a t) + N(y_0 + b t)
$$

which simplifies to:

$$
0 = C + Mx_0 + Ny_0 + t (Ma + Nb)
$$

Because this must hold for all values of t, two conditions emerge:

- The coefficient of t must be zero:

$$
Ma + Nb = 0
$$

- The constant term must be zero:

$$
C + Mx_0 + Ny_0 = 0
$$

These two equations are crucial in determining C, M, N.

---

4. Find the Relationship Between M, N, and the Line

From the first condition:

$$
Ma + Nb = 0
$$

This relates the slopes M, N to the direction vector (a, b):

$$
Ma = -N b
$$

or

$$
M = -\frac{N b}{a}
$$

if $a \neq 0$. If $a=0$, then the line is vertical in the x-direction, and special considerations apply.

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5. Expressing C in Terms of M and N

From the second condition:

$$
C = - Mx_0 - Ny_0
$$

Once M and N are related via the previous step, C can be computed accordingly.

Special Cases and Additional Considerations

Case 1: Line is Horizontal in XY-Plane

If the line is horizontal, i.e., $a=0$ and $b \neq 0$, then the parametric form reduces to:

$$
x = x_0, \quad y = y_0 + b t

\]

The condition \(Ma + Nb=0\) becomes:

\[
0 \times M + N \times b = 0 \Rightarrow N=0
\]

and:

\[
C + M x_0 + N y_0 = 0 \Rightarrow C + M x_0 = 0
\]

Thus, the plane equation reduces to:

\[
z = C + Mx
\]

with:

\[
C = - M x_0
\]

and \(N=0\).

Case 2: Line is Vertical in XY-Plane

If \(b=0\) and \(a \neq 0\), similar reasoning applies:

\[
Ma + Nb=0 \Rightarrow M a=0 \Rightarrow M=0
\]

and:

\[
C + N y_0=0
\]

with \(C= - N y_0\).

Constructing the Final Equation of the Plane

Once the parameters M, N, C are determined based on the line's properties, the plane's equation in the desired form is:

$$ z = C + Mx + Ny $$

This equation describes all points (x, y, z) lying on the plane that contains the line in the xy-plane.

Practical Example

Suppose you are given:

- Line in the xy-plane: passing through $(2, 3, 0)$ with direction vector $(4, -2)$.

Step 1: Write the parametric equations:

$$
x = 2 + 4 t, \quad y = 3 - 2 t
$$

Step 2: Set up the conditions:

$$
0 = C + M(2 + 4 t) + N(3 - 2 t)
$$

which simplifies to:

$$
0 = C + 2 M + 3 N + t (4 M - 2 N)
$$

For this to hold for all t:

$$
4 M - 2 N = 0 \Rightarrow 2 M = N
$$

and

\[
C + 2 M +

Frequently Asked Questions

How do I find the equation of a plane in the form Z = C + Mx + Ny that contains a given line in the XY-plane?

To find such a plane, first identify the line's parametric equations or points, then find its direction vector and a point on the line. Use these to determine the normal vector of the plane and its intercept C, ensuring the plane contains the entire line.

What information is needed to derive the equation of a plane containing a specific XY-line?

You need at least one point on the line and the direction vector of the line in the XY-plane. These help in calculating the plane's normal vector and the constants in the equation Z = C + Mx + Ny.

How does the line in the XY-plane influence the form of the plane equation Z = C + Mx + Ny?

Since the line lies entirely in the XY-plane (where Z=0), the plane must pass through all points of the line with Z=0. This condition helps determine the constants C, M, and N in the plane equation.

Can you provide a step-by-step method to find the plane equation containing a line in the XY-plane?

Yes. First, identify a point on the line and its direction vector. Next, assume Z=0 for the line points and substitute into the plane equation Z = C + Mx + Ny. Use the known points to solve for C, M, and N.

What are common mistakes to avoid when deriving the plane equation containing a given XY-line?

Avoid assuming the plane is horizontal or vertical without verification; ensure the entire line satisfies the plane equation; and be careful with the signs and substitutions during calculations.

How does the equation of the line in XY-plane help in determining the coefficients M and N in the plane equation?

The direction vector of the line provides the ratios of M and N, since the plane must contain all points of the line, allowing you to relate the line's slope to the coefficients M and N.

Additional Resources

Find The Equation Of The Plane In The Form Z = C + Mx + Ny Containing The Line In The XY-Plane Where

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Introduction

In the realm of three-dimensional geometry, the concept of planes is fundamental. Planes are flat, two-dimensional surfaces that extend infinitely in all directions within space. One of the essential tasks in analytic geometry is to determine the equation of a plane that satisfies certain conditions, such as passing through a specific line or point. Among the various forms of expressing a plane's equation, the form:

$$ Z = C + Mx + Ny $$

is particularly useful because it explicitly relates the z-coordinate to x and y coordinates, making it intuitive and straightforward for problems involving surfaces that can be expressed as functions of x and y.

In this article, we delve into the problem of determining the equation of a plane in the form $Z = C + Mx + Ny$, especially when the plane contains a given line lying in the XY-plane. This scenario is common in geometric modeling and engineering applications where constraints are specified along certain lines or curves.

Understanding the Basic Concepts

The Equation of a Plane in Different Forms

Planes in three-dimensional space can be represented in various forms, each suited to different problem contexts:

- General form: $Ax + By + Cz + D = 0$
- Intercept form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
- Explicit form (z as a function): $Z = C + Mx + Ny$

The explicit form $Z = C + Mx + Ny$ is particularly convenient when the plane can be expressed as a function $z = f(x, y)$. Here:

- C is the intercept of the plane with the z-axis.
- M and N are the partial derivatives of z with respect to x and y, respectively, or the slopes in those directions.

The XY-Plane and Lines in It

The XY-plane is characterized by $z=0$. A line lying in the XY-plane is thus described by equations involving only x and y (with z explicitly zero). For example, a line in the XY-plane can be represented parametrically:

\[
\begin{cases}
x = x_0 + t \cdot a \\
y = y_0 + t \cdot b \\

z = 0
\end{cases}
\]

where (x_0, y_0) is a point on the line and (a, b) is the direction vector.

The Problem Setup

Given:

- A line L lying entirely in the XY-plane.
- The desire to find a plane in the form $Z = C + Mx + Ny$.

The key question: What additional information is needed? Typically, to define such a plane, you need:

- The line L (point and direction).
- A point through which the plane passes (or the plane's intercepts).
- Or conditions such as the plane's slope or intercepts.

Step-by-Step Approach to Finding the Equation of the Plane

1. Identifying the Line in the XY-Plane

Suppose the line L in the XY-plane is given parametrically:

$$
x = x_1 + t a, \quad y = y_1 + t b, \quad z=0
$$

or in symmetric form:

$$
\frac{x - x_1}{a} = \frac{y - y_1}{b}
$$

with (x_1, y_1) a known point on the line, and (a, b) the direction vector.

Key Point: Since the line lies entirely in the XY-plane, its z-coordinate is zero, and any point on the line satisfies $z=0$.

2. Expressing the Plane in the Desired Form

We seek a plane:

$$
z = C + Mx + Ny
$$

which passes through the line L. For the plane to contain L, every point $(x, y, 0)$ on L must satisfy the plane's equation:

$$
0 = C + Mx + Ny
$$

for all points (x, y) on L.

3. Deriving Conditions from the Line

Since the line is in the XY-plane and the plane contains it, the entire line must satisfy:

$$
C + Mx + Ny = 0
$$

for all points (x, y) on L.

- If the line is infinite, then the entire line's points satisfy the equation, implying that the equation $C + Mx + Ny = 0$ defines the line itself in the XY-plane.

- If only a specific point (x_0, y_0) on the line is known, and the line's direction vector $\vec{d} = (a, b)$, then the plane must satisfy:

$$
0 = C + Mx_0 + Ny_0
$$

and for the entire line:

$$
0 = C + M(x_0 + a t) + N(y_0 + b t)
$$

which simplifies to:

$$
0 = (C + Mx_0 + Ny_0) + t (M a + N b)
$$

Since this must hold for all t, the coefficients must satisfy:

$$
\boxed{
\begin{cases}
C + Mx_0 + Ny_0 = 0 \\
M a + N b = 0
\end{cases}
}
$$

This is a system of equations that relates the plane's parameters to the line's data.

4. Solving for Plane Parameters

The key is to find C, M, N such that the above conditions hold.

- The second equation:

$$
M a + N b = 0
$$

relates the slopes M, N to the line's direction.

- The first equation:

$$
C = - (M x_0 + N y_0)
$$

connects C to M, N and the point (x_0, y_0).

Note: Because C depends on M and N, the problem reduces to choosing M and N satisfying the second equation, then calculating C.

Analytical Solution and Examples

Example 1: A Line with Known Point and Direction

Suppose the line passes through $(x_1, y_1) = (2, 3)$ with direction vector $(a, b) = (4, -2)$.

- The second condition:

$$
M \cdot 4 + N \cdot (-2) = 0 \Rightarrow 4 M - 2 N = 0
$$

which simplifies to:

$$
2 M = N
$$

- The first condition:

$$
C = - (M \cdot 2 + N \cdot 3) = - (2 M + 3 N)
$$

Substituting $N = 2 M$:

$$
C = - (2 M + 3 \times 2 M) = - (2 M + 6 M) = - 8 M
$$

Thus, the plane's equation becomes:

$$
z = C + M x + N y = -8 M + M x + 2 M y
$$

or, factoring out M:

$$
z = M (x + 2 y - 8)
$$

Since M is arbitrary, the plane is not uniquely determined; it depends on the choice of M.

Interpretation: The plane contains the line, and the parameter M can be chosen freely, corresponding to the slope in the z-direction relative to x and y.

Example 2: Specifying the Intercept C

If, for example, the plane passes through the point (x_0, y_0, z_0), then:

$$
z_0 = C + M x_0 + N y_0
$$

Using previous relations, and choosing specific M, N, you can determine C.

Geometric Interpretation and Insights

Relationship Between the Line and the Plane

The critical insight is that the line in the XY-plane constrains the plane's parameters:

- The plane must contain all points $(x, y, 0)$ satisfying $C + Mx + Ny=0$.

- For the entire line to lie in the plane, the line's parametric equations must satisfy this relation for all t.

- This leads to the two key equations relating M, N, C.

The Role of the Parameters C, M, N

- C: Determines the vertical shift of the plane relative to the origin.

- \(M, N \): Control the slopes of the plane in the x and y directions, respectively.

- The parametric relationship ensures the plane contains the entire line in the XY

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