Find The Flux Of The Vector Field F = 2xi - 3yj Across The Semi-circle R(t) = (cost)i + (sint)j.O St

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Understanding the Problem: Flux of a Vector Field Across a Semi-circle

When studying vector calculus, one common problem involves calculating the flux of a vector field across a specified surface or curve. In this scenario, our goal is to determine the flux of the vector field F = 2x□i - 3y□j across a semi-circular curve defined parametrically by R(t) = (cos□t)i + (sin□t)j, with t ranging over a specific interval.

Flux essentially measures how much of the field "flows" through a surface or along a curve. It is a fundamental concept in physics and engineering, especially in electromagnetism, fluid dynamics, and related fields.

Key Concepts and Definitions

1. Vector Field

A vector field assigns a vector to every point in space. In our case, the vector field is:

$$
\mathbf{F}(x, y) = 2x\, \mathbf{i} - 3y\, \mathbf{j}
$$

This field has components that depend on the coordinates x and y.

2. Flux

Flux (Φ) of a vector field across a curve or surface quantifies the number of field lines passing through it. For a curve in a plane, the flux can be calculated as a line integral involving the vector field and the differential element of the curve.

Mathematically, the flux across a curve C is:

$$
\Phi = \int_C \mathbf{F} \cdot \mathbf{n}\, ds
$$

where $\mathbf{n}$ is the unit normal vector to the curve, and ds is the differential arc length.

Alternatively, if the curve is oriented and the vector field is known, the flux can be computed as:

$$
\Phi = \int_C \mathbf{F} \cdot \mathbf{T}\, ds
$$

where $\mathbf{T}$ is the unit tangent vector, but in flux calculations, we often prefer the normal component.

3. Parameterization of the Curve

The given parametric curve:

$$
\mathbf{R}(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j}
$$

represents a circle of radius 1 centered at the origin. The interval of t defines which part of the circle we're considering.

Defining the Semi-circle and Its Parameterization

1. The Semi-circle Range

Depending on the problem context, the semi-circle could be the upper or lower half of the circle:

- Upper semi-circle: $t \in [0, \pi]$
- Lower semi-circle: $t \in [\pi, 2\pi]$

For the purpose of this calculation, let's assume the semi-circle is the upper half, with $t \in [0, \pi]$.

2. Parameterization Details

Using the parametric form:

$$
x(t) = \cos t, \quad y(t) = \sin t
$$

The derivatives are:

$$
x'(t) = -\sin t, \quad y'(t) = \cos t
$$

The differential arc length:

$$
ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt = \sqrt{\sin^2 t + \cos^2 t} dt = dt
$$

since $\sin^2 t + \cos^2 t = 1$.

Calculating the Flux: Step-by-Step Approach

1. Determine the Normal Vector

To compute flux, we need the outward (or inward) unit normal vector $\mathbf{n}$ to the curve at each point.

For a parameterized curve:

$$
\mathbf{R}(t) = x(t)\, \mathbf{i} + y(t)\, \mathbf{j}
$$

the tangent vector is:

$$
\mathbf{T}(t) = x'(t)\, \mathbf{i} + y'(t)\, \mathbf{j} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j}
$$

The unit tangent vector:

$$
\mathbf{\hat{T}}(t) = \mathbf{T}(t) \quad \text{(since } |\mathbf{T}|=1\text{)}
$$

The outward normal vector $\mathbf{n}(t)$ is perpendicular to $\mathbf{\hat{T}}(t)$. For a circle, the outward normal points radially outward:

$$
\mathbf{n}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j}
$$

which coincides with the position vector normalized (since the circle has radius 1).

2. Express the Field $\mathbf{F}$ Along the Curve

At each point:

$$
x = \cos t, \quad y = \sin t
$$

$$
\mathbf{F}(x, y) = 2x\, \mathbf{i} - 3y\, \mathbf{j} = 2 \cos t\, \mathbf{i} - 3 \sin t\, \mathbf{j}
$$

3. Compute $\mathbf{F} \cdot \mathbf{n}$

$$
\mathbf{F} \cdot \mathbf{n} = (2 \cos t)(\cos t) + (-3 \sin t)(\sin t) = 2 \cos^2 t - 3 \sin^2 t
$$

4. Set Up the Line Integral for Flux

The flux across the semi-circle is:

$$
\Phi = \int_{t=a}^{b} \mathbf{F} \cdot \mathbf{n} \, ds
$$

Since $ds = dt$, the integral becomes:

$$
\Phi = \int_{t=a}^{b} \left(2 \cos^2 t - 3 \sin^2 t \right) dt
$$

For the upper semi-circle:

$$
a=0, \quad b=\pi
$$

Calculating the Integral: Detailed Steps

1. Simplify the integrand

Express the integrand in terms of double angles to facilitate integration:

$$
2 \cos^2 t - 3 \sin^2 t
$$

Recall the identities:

$$
\cos^2 t = \frac{1 + \cos 2t}{2}
$$

$$
\sin^2 t = \frac{1 - \cos 2t}{2}
$$

\]

Substitute:

$$
2 \times \frac{1 + \cos 2t}{2} - 3 \times \frac{1 - \cos 2t}{2} = (1 + \cos 2t) - \frac{3}{2}(1 - \cos 2t)
$$

Distribute:

$$
1 + \cos 2t - \frac{3}{2} + \frac{3}{2} \cos 2t
$$

Combine like terms:

$$
\left(1 - \frac{3}{2}\right) + \left(\cos 2t + \frac{3}{2} \cos 2t\right) = -\frac{1}{2} + \frac{5}{2} \cos 2t
$$

Thus, the integrand simplifies to:

$$
-\frac{1}{2} + \frac{5}{2} \cos 2t
$$

---

2. Integrate over $t \in [0, \pi]$

The flux:

$$
\Phi = \int_0^{\pi} \left(-\frac{1}{2} + \frac{5}{2} \cos 2t \right) dt
$$

Break into two integrals:

$$
\Phi = -\frac{1}{2} \int_0^{\pi} dt + \frac{5}{2} \int_0^{\pi} \cos 2t\, dt
$$

Calculate each:

- First integral:

$$
-\frac{1}{2} \times (\pi - 0) = -\frac{\pi}{2}
$$

- Second integral:

$$
\int \cos 2t\, dt = \frac{\sin 2t}{2}
$$

Evaluated from 0 to π:

$$
\frac{\sin 2t}{2} \Big|_0^{\pi} = \frac{\sin 2\pi - \sin 0}{2} = \frac{0 - 0}{2} = 0
$$

Therefore, the total flux:

\[

\Phi = -

Frequently Asked Questions

What is the vector field F given in the problem?

The vector field is F = 2x i - 3y j.

What is the parameterization of the semi-circle R(t)?

The semi-circle is parameterized as R(t) = (cos t) i + (sin t) j, typically for t in [0, □].

How do we interpret the flux of a vector field across a semi-circular boundary?

The flux represents the amount of the vector field passing outward through the boundary surface—in this case, the semi-circle—calculated via a line integral of the vector field's normal component.

What is the method to compute the flux of F across the semi-circle?

The flux is computed using the line integral of F · n ds over the boundary, where n is the outward normal vector and ds is the differential arc length.

Should we use Green's theorem to find the flux in this scenario?

Green's theorem relates a line integral around a simple closed curve to a double integral over the region, but since the semi-circle is not closed, we need to consider the boundary and possibly complete the curve or use the divergence theorem if applicable.

How do we find the outward normal vector n(t) for the semi-circle?

For the semi-circle parameterized as R(t) = (cos t, sin t), the outward normal vector at each point is n(t) = (cos t, sin t), which points outward from the circle.

What is the divergence of the vector field F?

The divergence of F = 2x i - 3y j is div F = □(2x)/□x + □(-3y)/□y = 2 - 3 = -1.

How does divergence relate to flux in this problem?

By the divergence theorem, the flux of F across the closed boundary of a region equals the double integral of div F over the region. Since we're dealing with a semi-circle, we can consider the full circle or adapt the theorem accordingly to find the flux.

Additional Resources

Find The Flux Of The Vector Field F = 2xi - 3yj Across The Semi-circle R(t) = (cos t)i + (sin t)j, 0 □ t □ □

Understanding and calculating the flux of a vector field across a given surface or curve is a fundamental concept in vector calculus, with applications spanning physics, engineering, and computer graphics. In this article, we explore the process of finding the flux of the vector field F = 2xi - 3yj across a semi-circular arc defined parametrically by R(t) = (cos t)i + (sin t)j, for t ranging from 0 to □. We will delve into the theoretical background, step-by-step calculation procedures, and interpret the results to provide a comprehensive understanding.

Introduction to Flux in Vector Calculus

Flux measures how much of a vector field passes through a surface or along a curve. In simpler terms, it quantifies the flow or transfer of a quantity represented by the vector field through a specified surface or boundary.

For a vector field F and a surface S, the flux Φ is given by the surface integral:

$$
\Phi = \iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS
$$

where n is the unit normal vector to the surface. When dealing with curves, especially in two dimensions, the flux across a curve C is computed via a line integral of the form:

$$
\Phi = \int_{C} \mathbf{F} \cdot \mathbf{n}\, ds
$$

Here, n is the outward normal vector to the boundary curve, and ds is the differential arc length.

Understanding the Vector Field and the Curve

The Vector Field: F = 2xi - 3yj

The given vector field:

$$
\mathbf{F}(x, y) = 2x\, \mathbf{i} - 3y\, \mathbf{j}
$$

is a linear field where the components depend linearly on x and y. The field points outward in the x-direction when x is positive and inward when x is negative, similarly for y. Its divergence, which indicates the net flow out of a small region, is:

$$
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3y) = 2 - 3 = -1
$$

A negative divergence suggests the field is generally converging inward or acting like a sink.

The Curve: Semi-circle R(t) = (cos t)i + (sin t)j, 0 □ t □ □

This parametrization describes a semi-circular arc of radius 1 centered at the origin, starting at (1, 0) for t=0 and ending at (-1, 0) for t=□. The curve lies in the upper half-plane and is oriented counterclockwise.

Setting Up the Flux Integral

To compute the flux of F across the semi-circle, we need to:

- Determine the normal vector n at each point on the curve.
- Compute the differential arc length ds.
- Express the vector field F along the curve R(t).

Choosing the Normal Vector

Since the curve is a semi-circle, the outward normal vector at each point can be obtained by differentiating $R(t)$ and then finding a perpendicular vector pointing outward.

Given:

$$
\mathbf{R}(t) = (\cos t, \sin t)
$$

The derivative:

$$
\mathbf{R}'(t) = (-\sin t, \cos t)
$$

The tangent vector at t is $R'(t)$, and a unit normal vector $n(t)$ can be obtained by rotating $R'(t)$ by 90 degrees. For the outward normal, considering the semi-circle in the upper half-plane, the outward normal at each point is simply:

$$
\mathbf{n}(t) = (\cos t, \sin t)
$$

which coincides with the position vector normalized (since radius = 1). Alternatively, for a circle centered at the origin, the outward normal at a point on the boundary is just the position vector normalized, which in this case is already a unit vector.

Calculating the Flux

Step 1: Expressing the Vector Field Along the Curve

Substitute the parametrization into F:

$$
\mathbf{F}(R(t)) = 2 \cos t\, \mathbf{i} - 3 \sin t\, \mathbf{j}
$$

Step 2: Dot Product with the Normal Vector

The outward normal vector:

$$
\mathbf{n}(t) = (\cos t, \sin t)
$$

The dot product:

$$
\mathbf{F}(R(t)) \cdot \mathbf{n}(t) = (2 \cos t) \cdot \cos t + (-3 \sin t) \cdot \sin t = 2 \cos^2 t - 3 \sin^2 t
$$

Step 3: Expressing ds and setting up the integral

The differential arc length:

$$
ds = |\mathbf{R}'(t)| dt
$$

Given:

$$
|\mathbf{R}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1
$$

Thus,

$$
ds = dt
$$

The limits of integration are from $t = 0$ to $t = \pi$. The flux □ is:

$$
□ = \int_{0}^{\pi} \left(2 \cos^2 t - 3 \sin^2 t \right) dt
$$

Performing the Integration

Using Trigonometric Identities

Recall the identities:

$$
\cos^2 t = \frac{1 + \cos 2t}{2}
$$

$$
\sin^2 t = \frac{1 - \cos 2t}{2}
$$

Rewrite the integral:

$$
\square = \int_0^{\pi} \left[2 \times \frac{1 + \cos 2t}{2} - 3 \times \frac{1 - \cos 2t}{2} \right] dt
$$

Simplify:

$$
\square = \int_0^{\pi} \left[(1 + \cos 2t) - \frac{3}{2} (1 - \cos 2t) \right] dt
$$

Distribute:

$$
\square = \int_0^{\pi} \left[1 + \cos 2t - \frac{3}{2} + \frac{3}{2} \cos 2t \right] dt
$$

\]

Combine like terms:

$$
= \int_0^{\pi} \left[(1 - \frac{3}{2}) + (\cos 2t + \frac{3}{2} \cos 2t) \right] dt
$$

$$
= \int_0^{\pi} \left[-\frac{1}{2} + \left(1 + \frac{3}{2}\right) \cos 2t \right] dt
$$

$$
= \int_0^{\pi} \left[-\frac{1}{2} + \frac{5}{2} \cos 2t \right] dt
$$

Step 4: Integrate Term-by-Term

Integrate:

$$
= \left[-\frac{1}{2} t + \frac{5}{2} \times \frac{\sin 2t}{2} \right]_0^{\pi}
$$

Simplify:

$$
= \left[-\frac{1}{2} t + \frac{5}{4} \sin 2t \right]_0^{\pi}
$$

Evaluate at the bounds:

At t = □:

$$
-\frac{1}{2} \pi + \frac{5}{4} \sin 2\pi = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}
$$

At t = 0:

$$
-\frac{1}{2} \times 0 + \frac{5}{4} \times \sin 0 = 0 + 0 = 0
$$

Subtract:

$$
\square = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}
$$

Final Result:

$$
\boxed{
\text{Flux } \Phi = -\frac{\pi}{2}
}
$$

Interpretation of the Result

The negative sign indicates that the net flow of the vector field F across the semi-circular boundary is directed inward relative to the chosen outward normal. Physically,

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