Find The Matrix Of The Orthogonal Projection In R 2 Onto The Line X1 = 2x2. Hint: What Is The Matrix

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Understanding how to find the matrix of an orthogonal projection in $\mathbb{R}^2$ onto a specific line is fundamental in linear algebra. Specifically, projecting vectors onto a line such as $X_1 = 2X_2$ involves deriving a matrix that, when multiplied by any vector in $\mathbb{R}^2$, yields its orthogonal projection onto that line. This article guides you through the process step-by-step, explaining the concepts, the methodology, and providing the explicit matrix form for this particular projection.

Introduction to Orthogonal Projection in $\mathbb{R}^2$

What is Orthogonal Projection?

Orthogonal projection is a linear transformation that maps any vector in a vector space onto a subspace such in a way that the difference between the original vector and its projection is orthogonal to that subspace. For a line in $\mathbb{R}^2$, the orthogonal projection of a vector onto that line gives the closest point on the line to the original vector.

Why Find the Projection Matrix?

The projection matrix simplifies the process of projecting multiple vectors without recalculating the projection formula each time. Once the matrix is found, projecting any vector onto the line is as simple as matrix-vector multiplication.

Understanding the Line $X_1 = 2X_2$

Representing the Line in Vector Form

The line $X_1 = 2X_2$ can be expressed parametrically by choosing a parameter t:

$$
\begin{cases}
X_1 = 2t \\
X_2 = t
\end{cases}
$$

\]
In vector form:
\[
\mathbf{v}(t) = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]
This shows that the line passes through the origin and is directed along the vector \(\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\).

The Direction Vector \(\mathbf{u}\)

The vector \(\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\) serves as a basis vector for the line. Any point on the line can be written as a scalar multiple of \(\mathbf{u}\).

Finding the Projection Matrix onto the Line

Step 1: Normalize the Direction Vector

To find the orthogonal projection matrix, we often start with the direction vector \(\mathbf{u}\). First, calculate its magnitude:
\[
\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}
\]
The unit vector in the direction of \(\mathbf{u}\) is:
\[
\mathbf{\hat{u}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Step 2: Recall the Projection Formula

The orthogonal projection of a vector \(\mathbf{x}\) onto a unit vector \(\mathbf{\hat{u}}\) is given by:
\[
\operatorname{proj}_{\mathbf{\hat{u}}}(\mathbf{x}) = (\mathbf{\hat{u}} \mathbf{\hat{u}}^T) \mathbf{x}
\]
which leads to a projection matrix:
\[
P = \mathbf{\hat{u}} \mathbf{\hat{u}}^T
\]

Step 3: Compute the Projection Matrix \(P\)

Substitute \(\mathbf{\hat{u}}\) into the formula:
\[
P = \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(

\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \end{bmatrix} \right) = \frac{1}{5}
\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}
\]
Performing the matrix multiplication:
\[
P = \frac{1}{5} \begin{bmatrix}
2 \times 2 & 2 \times 1 \\
1 \times 2 & 1 \times 1
\end{bmatrix} = \frac{1}{5} \begin{bmatrix}
4 & 2 \\
2 & 1
\end{bmatrix}
\]

Explicit Form of the Projection Matrix

Final Projection Matrix \(P\)

Based on the calculation above, the matrix of the orthogonal projection onto the line \(X_1 = 2X_2\) in \(\mathbb{R}^2\) is:
\[
\boxed{
P = \frac{1}{5} \begin{bmatrix}
4 & 2 \\
2 & 1
\end{bmatrix}
}
\]
This matrix, when multiplied by any vector \(\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\), produces the orthogonal projection of \(\mathbf{x}\) onto the line.

Applications and Importance of the Projection Matrix

Use in Data Analysis and Computer Graphics

Projection matrices are fundamental in various fields such as data analysis, computer graphics, and machine learning. They allow for efficient data reduction, visualization, and transformations onto specific subspaces.

Projection in Least Squares and Regression

In regression analysis, projection matrices help in understanding the least squares solution, where the goal is to project data points onto a subspace spanned by explanatory variables.

Summary and Key Takeaways

- The line $X_1 = 2X_2$ in $\mathbb{R}^2$ can be represented by the direction vector $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- The orthogonal projection matrix onto this line is derived using the normalized direction vector.
- The final matrix is $P = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.
- Applying P to any vector in $\mathbb{R}^2$ gives its orthogonal projection onto the line $X_1 = 2X_2$.

Conclusion

Finding the matrix of the orthogonal projection in $\mathbb{R}^2$ onto a specific line, such as $X_1 = 2X_2$, involves understanding the line's direction vector and applying the projection formula. The process is straightforward once the direction vector is identified and normalized. The resulting matrix enables efficient projection of vectors, with applications spanning numerous disciplines in science and engineering. Whether used in data analysis, computer graphics, or solving systems of equations, the projection matrix is a vital tool in linear algebra.

Frequently Asked Questions

What is the goal when finding the matrix of the orthogonal projection in R^2 onto the line $x_1 = 2x_2$?

The goal is to find a matrix that, when multiplied by any vector in R^2, projects it orthogonally onto the line defined by $x_1 = 2x_2$.

How do we represent the line $x_1 = 2x_2$ in vector form?

The line can be represented parametrically as $(x_1, x_2) = t(2, 1)$ for all real t.

What is the direction vector for the line $x_1 = 2x_2$?

The direction vector for the line is v = (2, 1).

What is the general formula for the orthogonal projection matrix onto a line with direction vector v?

The projection matrix P = (v v^T) / (v^T v), where v is the direction vector.

How do we compute the projection matrix for v = (2, 1)?

Calculate P = (v v^T) / (v^T v): P = ((2, 1)^T (2, 1)) / ($2^2 + 1^2$) = ([2;1][2,1]) / 5.

What is the explicit matrix form of the projection onto the line $x_1 = 2x_2$?

The projection matrix is P = (1/5) [[4, 2], [2, 1]].

How can we verify that the matrix P is a projection matrix?

Check that $P^2 = P$ and that P is symmetric; these are properties of orthogonal projection matrices.

What is the final answer for the matrix of the orthogonal projection onto the line $x_1 = 2x_2$?

The matrix is P = (1/5) [[4, 2], [2, 1]].

Additional Resources

Find The Matrix Of The Orthogonal Projection In R^2 Onto The Line $X_1 = 2X_2$: A Detailed Exploration

Understanding how to find the matrix of an orthogonal projection onto a specific line in \(\mathbb{R}^2\) is fundamental in linear algebra, especially in applications involving projections, least squares, and geometric interpretations. The problem at hand—finding the matrix of the orthogonal projection in \(\mathbb{R}^2\) onto the line defined by the equation \(X_1 = 2X_2\)—serves as an excellent example to elucidate the concepts of projections, matrices, and their derivations. This article aims to provide a comprehensive, step-by-step guide to solving this problem, including theoretical background, methods, and practical insights.

Understanding Orthogonal Projections in

$\mathbb{R}^2$

Before diving into the derivation, it is crucial to understand what an orthogonal projection is and how it functions geometrically and algebraically.

What Is an Orthogonal Projection?

An orthogonal projection of a vector $\mathbf{v}$ onto a line (or subspace) L in $\mathbb{R}^2$ is the vector in L that is closest to $\mathbf{v}$, minimizing the Euclidean distance between $\mathbf{v}$ and the line. Geometrically, this is akin to dropping a perpendicular from $\mathbf{v}$ onto the line L.

Mathematically, the orthogonal projection $\operatorname{proj}_L(\mathbf{v})$ can be expressed as:

$$
\operatorname{proj}_L(\mathbf{v}) = \mathbf{p} \quad \text{such that} \quad \mathbf{v} - \mathbf{p} \perp L
$$

This projection is linear, meaning it can be represented by a matrix P such that:

$$
\operatorname{proj}_L(\mathbf{v}) = P \mathbf{v}
$$

The Line $X_1 = 2X_2$: Geometric and Algebraic Representation

The line given by $X_1 = 2X_2$ can be interpreted geometrically as passing through the origin with a specific slope, or as the span of a particular vector.

Parametric Equation of the Line

Since $X_1 = 2X_2$, any point $\mathbf{x} = (x_1, x_2)$ on the line satisfies:

$$
x_1 = 2x_2
$$

Choosing x_2 as a free parameter t, the parametric form is:

$$
\mathbf{x} = (2t, t) = t(2,1)
$$

Thus, the line L is the span of the vector:

$$
\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

The line consists of all scalar multiples of $\mathbf{u}$.

Deriving the Orthogonal Projection Matrix P

Given the line represented by a non-zero vector $\mathbf{u}$, the orthogonal projection matrix onto the span of $\mathbf{u}$ is well-understood in linear algebra.

Formula for Projection onto a Vector

If $\mathbf{u} \neq \mathbf{0}$, then the orthogonal projection matrix P onto the line spanned by $\mathbf{u}$ is:

$$
P = \frac{\mathbf{u} \mathbf{u}^\top}{\mathbf{u}^\top \mathbf{u}}
$$

where:

- $\mathbf{u} \mathbf{u}^\top$ is the outer product of $\mathbf{u}$ with itself, resulting in a 2×2 matrix.
- $\mathbf{u}^\top \mathbf{u}$ is the dot product (a scalar).

Calculating the Components

Given $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

- Dot product:

$$
\mathbf{u}^\top \mathbf{u} = 2^2 + 1^2 = 4 + 1 = 5
$$

- Outer product:

$$
\mathbf{u} \mathbf{u}^\top = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}
$$

Thus, the projection matrix P is:

$$
P = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}
$$

which simplifies to:

$$
P = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}
$$

Verifying the Projection Matrix

Verification is crucial to ensure the derived matrix performs as expected.

Properties of a Projection Matrix

A matrix P that projects onto a line should satisfy:

- Idempotency: $P^2 = P$
- Symmetry: $P^\top = P$

Let's verify these.

Idempotency Check

Compute P^2:

$$
P^2 = \left(\frac{\mathbf{u} \mathbf{u}^\top}{\mathbf{u}^\top \mathbf{u}} \right)^2 = \frac{\mathbf{u} \mathbf{u}^\top \mathbf{u} \mathbf{u}^\top}{(\mathbf{u}^\top \mathbf{u})^2}
$$

Note that:

$$
\mathbf{u}^\top \mathbf{u} = 5
$$

and:

$$
\mathbf{u}^\top \mathbf{u} \mathbf{u}^\top = 5 \mathbf{u}^\top
$$

Thus:

$$
P^2 = \frac{\mathbf{u} (\mathbf{u}^\top \mathbf{u}) \mathbf{u}^\top}{(\mathbf{u}^\top \mathbf{u})^2 } = \frac{\mathbf{u} \times 5 \times \mathbf{u}^\top}{25} = \frac{5}{25} \mathbf{u} \mathbf{u}^\top = \frac{1}{5} \mathbf{u} \mathbf{u}^\top = P
$$

Confirmed: $P^2 = P$.

Symmetry Check

Since:

$$
P^\top = \left(\frac{\mathbf{u} \mathbf{u}^\top}{\mathbf{u}^\top \mathbf{u}} \right)^\top = \frac{(\mathbf{u} \mathbf{u}^\top)^\top}{\mathbf{u}^\top \mathbf{u}} = \frac{\mathbf{u} \mathbf{u}^\top}{\mathbf{u}^\top \mathbf{u}} = P
$$

which confirms symmetry.

Features, Pros, and Cons of the Projection Matrix

Features:

- Linear transformation: P is a linear operator that projects vectors onto a specific line.
- Symmetric and idempotent: Ensures the projection is consistent and stable.
- Easy to compute: Based on the outer product of the direction vector.

Pros:

- Simple derivation: The formula relies on basic vector operations.
- Computational efficiency: Suitable for algorithms requiring repeated projections.

- Geometric clarity: Directly relates to the span of a vector.

Cons:

- Limited to linear subspaces: Only applicable to linear subspaces like lines and planes.
- Requires non-zero vector: The method hinges on the direction vector being non-zero.
- No orthogonal complement information: Focuses solely on the projection, not on the orthogonal complement.

Practical Applications and Extensions

The matrix obtained isn't just an academic exercise; it plays a key role in various practical scenarios:

- Least squares problems: When fitting data points to a line, the projection matrix helps in computing residuals.
- Computer graphics: For rendering and geometric transformations.
- Signal processing: Filtering signals onto subspaces.

Furthermore, the method generalizes to higher dimensions and different subspaces, such as projecting onto planes in $\mathbb{R}^3$ or subspaces defined by matrices.

Summary and Final Remarks

In this detailed exploration, we've shown that finding the matrix of the orthogonal projection onto the line \(X_1 = 2X_

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