Find The Power Series Representation For G Centered At 0 By Differentiating Or Integrating The Power

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Understanding how to find the power series representation of a function centered at zero (also known as Maclaurin series) is a fundamental skill in calculus and mathematical analysis. When dealing with a function G, often we are interested in expressing it as an infinite sum of powers of x, which can simplify calculations, enable approximations, and provide insight into the function's behavior near the origin. One powerful approach to deriving these series representations involves differentiating or integrating known power series, leveraging the properties of convergence and term-by-term operations.

This article guides you through the process of finding the power series representation for a function G centered at 0 by differentiating or integrating existing power series. We will explore the theoretical foundations, step-by-step procedures, and illustrative examples to solidify your understanding.

Understanding Power Series and Centering at Zero (Maclaurin Series)

Before delving into differentiation and integration techniques, it's crucial to clarify what a power series centered at zero entails.

What Is a Power Series?

A power series is an infinite sum of the form:

$$
\sum_{n=0}^\infty a_n x^n
$$

where:
- a_n are coefficients,
- x is the variable,
- the series converges for x within some radius R.

Centering at Zero: The Maclaurin Series

When the series expansion is centered at 0, the series is called a Maclaurin series:

$$
G(x) = \sum_{n=0}^\infty a_n x^n
$$

This representation is particularly useful because it simplifies calculations and analyses near the origin.

Methods for Finding Power Series Representations

There are two main methods for deriving the power series for a function G:

- **Differentiation of Known Series:** Differentiating an existing power series to find the series for derivatives of G, then integrating back if necessary.

- **Integration of Known Series:** Integrating an existing power series to obtain the series for the integral of G, or vice versa.

Both methods rely on the principle that, within the radius of convergence, term-by-term differentiation and integration of a power series are valid operations.

Fundamental Theorems and Principles

Term-by-Term Differentiation and Integration

If a power series converges for $|x| < R$, then:

- Differentiating the series term-by-term yields a new series that also converges for $|x| < R$.

$$
\frac{d}{dx} \left(\sum_{n=0}^\infty a_n x^n \right) = \sum_{n=1}^\infty n a_n x^{n-1}
$$

- Integrating the series term-by-term yields:

$$
\int \left(\sum_{n=0}^\infty a_n x^n \right) dx = C + \sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1}
$$

where C is the constant of integration.

Radius of Convergence Preservation

The operations of differentiation and integration do not change the radius of convergence; they only affect the series' form within that radius, which makes these techniques highly effective.

Step-by-Step Approach to Finding Power Series for G

The process depends on what known information we have about G or related functions.

Scenario 1: Deriving G via Differentiating a Known Series

Suppose you know the power series for a function F, and G is related to F via differentiation:

$$
G(x) = \frac{d}{dx} F(x)
$$

Steps:

1. Identify the known series for $F(x)$:
Find the power series expansion:

$$
F(x) = \sum_{n=0}^\infty c_n x^n
$$

2. Differentiate term-by-term:

$$
G(x) = F'(x) = \sum_{n=1}^\infty n c_n x^{n-1}
$$

3. Rewrite to standard form:

$$
G(x) = \sum_{n=0}^\infty (n+1) c_{n+1} x^n
$$

4. Determine the coefficients a_n for G:

$$
a_n = (n+1) c_{n+1}
$$

5. Verify the radius of convergence to ensure the series is valid in the desired interval.

Example: Power Series for e^x and its Derivative

- Known series:

$$
e^x = \sum_{n=0}^\infty \frac{x^n}{n!}
$$

- Derivative:

$$
\frac{d}{dx} e^x = e^x
$$

- Deriving the power series for $G(x) = e^x$:

$$
G(x) = \sum_{n=1}^\infty \frac{n}{n!} x^{n-1} = \sum_{n=0}^\infty \frac{n+1}{(n+1)!} x^n = \sum_{n=0}^\infty \frac{1}{n!} x^n
$$

which confirms the series for e^x.

Scenario 2: Deriving G via Integrating a Known Series

Suppose G is an integral of a known function F:

$$

$$
G(x) = \int_0^x F(t) dt
$$

Steps:

1. Find the power series for $F(t)$:

$$
F(t) = \sum_{n=0}^\infty c_n t^n
$$

2. Integrate term-by-term:

$$
G(x) = C + \sum_{n=0}^\infty \frac{c_n}{n+1} x^{n+1}
$$

3. Determine the constant C:

Since the integral is from 0, $G(0) = 0$, so:

$$
G(0) = 0 \Rightarrow C = 0
$$

4. Express the series:

$$
G(x) = \sum_{n=0}^\infty \frac{c_n}{n+1} x^{n+1}
$$

5. Rewrite in standard form:

$$
G(x) = \sum_{n=1}^\infty \frac{c_{n-1}}{n} x^{n}
$$

Examples of Power Series via Differentiation and Integration

Example 1: Power Series for $\sin x$

- Known series:

$$
\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}
$$

- To find the power series for $\cos x$, differentiate $\sin x$:

$$
\cos x = \frac{d}{dx} \sin x
$$

- Differentiate term-by-term:

$$
\cos x = \sum_{n=0}^\infty (-1)^n \frac{(2n+1)}{(2n+1)!} x^{2n}
$$

- Simplify:

$$
\cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}
$$

which is the well-known power series for cosine.

Example 2: Power Series for $\arctan x$

- Known series:

$$
\arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}
$$

- To derive the power series for $\frac{1}{1 + x^2}$, differentiate $\arctan x$:

$$
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}
$$

- Differentiate term-by-term:

$$
\frac{1}{1 + x^2} = \sum_{n=0}^\infty (-1)^n x^{2n}
$$

- Confirming the geometric series expansion.

Practical Applications and Tips

- Check the radius of convergence: Always verify the interval where the series converges,

Frequently Asked Questions

How can I find the power series representation of a function G centered at 0 by differentiating or integrating its series?

To find the power series of G centered at 0, start with a known series expansion of a related function, then differentiate or integrate term-by-term as needed, ensuring the series converges within the desired radius.

What is the process for obtaining the power series of G by integrating a known series?

Integrate the known power series term-by-term, adjusting for constants of integration if necessary, to find the series representation of G. This is valid within the radius of convergence of the original series.

How does differentiating a power series help in finding the series for G?

Differentiating a power series term-by-term can help identify the series for G if G is related to the derivative of a known function. It can also simplify the process of finding G's series by reversing differentiation.

Can I differentiate or integrate a power series outside its radius of convergence?

No, differentiation and integration of power series are valid only within the radius of convergence. Outside this radius, the series may not represent the function accurately.

What are the key steps to find the power series of G centered at 0 using differentiation?

Identify a known series expansion, differentiate it term-by-term to relate to G, and then express G as a power series. Confirm the convergence radius remains valid after differentiation.

How do I handle constants of integration when integrating a power series to find G?

Constants of integration are added after integrating the series. To determine the constant, use initial conditions or known values of G at specific points.

Are there any common functions whose power series representations are used as starting points for differentiation or integration?

Yes, functions like e^x, sin x, cos x, and 1/(1 - x) have well-known power series that can be differentiated or integrated to find related series for G.

Additional Resources

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Introduction

Power series are fundamental tools in mathematical analysis, providing a way to represent functions as infinite sums of powers of a variable. They are especially useful for approximating functions, solving differential equations, and analyzing function behavior near specific points. When working with a function \(G(x) \), a common task is to find its power series expansion centered at a particular point—here, at 0. This process often involves differentiating or integrating existing power series representations to derive new series, especially when the function \(G \) is related to other functions with known series expansions.

In this comprehensive guide, we delve into the methods of obtaining the power series representation of a function \(G \) centered at 0, specifically by leveraging differentiation and integration techniques. We will explore the theoretical foundations, step-by-step procedures, and practical examples to equip you with a deep understanding of this process.

Understanding Power Series and Centering at 0

Before diving into differentiation and integration methods, it’s crucial to understand what it means to find a power series centered at 0:

- Power Series Definition: A function \(G(x) \) can be expressed as a power series centered at 0 if it has the form:

\[
G(x) = \sum_{n=0}^\infty a_n x^n

\]

where the coefficients a_n are constants, and the series converges within some radius of convergence R.

- Centering at 0 (Maclaurin Series): When centered at 0, the series is known as a Maclaurin series. It simplifies analysis because derivatives are evaluated at 0, and the series expansion captures the function's behavior near the origin.

Why Use Differentiation or Integration?

Often, the primary challenge is to determine the coefficients a_n. If the function G is complicated, directly expanding it into a power series may be difficult. Instead, it’s more manageable to:

- Use differentiation: If G is related to a known function with a known series, differentiating that series can produce a new series for G.

- Use integration: Similarly, integrating a known series can lead to the power series of functions that are integrals of known functions.

This approach leverages the properties of power series: term-by-term differentiation and integration are valid within the radius of convergence.

Step-by-Step Methods for Finding Power Series via Differentiation or Integration

1. Starting from a Known Series

The most straightforward method begins with a known power series expansion of a related function. For example:

$$
f(x) = \sum_{n=0}^\infty c_n x^n
$$

If $G(x)$ relates to $f(x)$ via differentiation or integration, then:

- Differentiation: $G(x) = f'(x)$
- Integration: $G(x) = \int f(x) \, dx$

By applying differentiation or integration to the known series, you can derive the series for $G(x)$.

---

2. Differentiating Power Series

Step 1: Write the known power series:

$$
f(x) = \sum_{n=0}^\infty c_n x^n
$$

Step 2: Differentiate term-by-term:

$$
f'(x) = \sum_{n=1}^\infty n c_n x^{n-1}
$$

Note: The $n=0$ term vanishes upon differentiation because its derivative is zero.

Step 3: Reindex to express in standard form:

$$
f'(x) = \sum_{n=0}^\infty (n+1) c_{n+1} x^{n}
$$

Step 4: Identify the coefficients a_n for $G(x) = f'(x)$:

$$
a_n = (n+1) c_{n+1}
$$

Key Point: This process allows you to determine the power series for G once you know the coefficients c_n.

---

3. Integrating Power Series

Step 1: Start with the known series:

$$
f(x) = \sum_{n=0}^\infty c_n x^n
$$

Step 2: Integrate term-by-term:

$$
\int f(x) \, dx = \sum_{n=0}^\infty c_n \frac{x^{n+1}}{n+1} + C
$$

where C is the constant of integration, which can be determined based on initial conditions or the value of the function at 0.

Step 3: Reindex the sum to express as a power series:

\[

\int f(x) \, dx = \sum_{n=1}^\infty c_{n-1} \frac{x^{n}}{n} + C
\]

Step 4: For the function \(G(x) \), the power series coefficients are:

\[
a_n = \frac{c_{n-1}}{n}
\]

This process is particularly useful when \(G \) is an antiderivative of a known function.

Practical Examples

To solidify understanding, let’s explore concrete examples illustrating how differentiation and integration help find the power series of \(G \).

Example 1: Deriving the Series for \(\sin x \) from the Geometric Series

Known Series: The geometric series for \(|x| < 1 \):

\[
\frac{1}{1 - x} = \sum_{n=0}^\infty x^n
\]

Goal: Find the power series for \(\sin x \) centered at 0.

Method:

- Recognize that \(\sin x \) can be expressed as a power series obtained by differentiating or integrating related functions.

- The derivative of \(-\cos x \) is \(\sin x \). If we find the series for \(\cos x \), then differentiate to find \(\sin x \).

Series for \(\cos x \):

\[
\cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}
\]

Differentiating:

\[
\frac{d}{dx} \cos x = - \sin x
\]

Thus,

$$
- \sin x = \frac{d}{dx} \left(\sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} \right) = \sum_{n=1}^\infty (-1)^n \frac{2n x^{2n-1}}{(2n)!}
$$

Rearranged, the series for $\sin x$:

$$
\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}
$$

Conclusion: By differentiating the series for $\cos x$, we obtained the series for $\sin x$.

Example 2: Finding the Power Series for $G(x) = \arctan x$

Known Series: The geometric expansion for $\frac{1}{1 + x^2}$:

$$
\frac{1}{1 + x^2} = \sum_{n=0}^\infty (-1)^n x^{2n}
$$

Goal: Find the power series for $\arctan x$.

Method:

- Recall that:

$$
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}
$$

- Therefore, $G(x) = \arctan x$ is an integral of $\frac{1}{1 + x^2}$.

Process:

$$
\arctan x = \int \frac{1}{1 + x^2} \, dx + C
$$

- Using the known series:

$$
\frac{1}{1 + x^2} = \sum_{n=0}^\infty (-1)^n x^{2n}
$$

- Integrate term-by-term:

$$
\arctan x = \int \left(\sum_{n=0}^\infty (-1)^n x^{2n} \right) dx + C =

\sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} + C
\]

- The constant \(C \) can be determined by evaluating at \(x=0 \):

\[
\arctan 0 = 0 \Rightarrow C=0
\]

Result:

\[
\boxed{
\arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}
}
\]

Radius of Convergence and Validity

When deriving power series via differentiation or integration, it’s essential to consider the radius of convergence:

- The radius of convergence for the original series determines

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