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Understanding how to solve differential equations with initial conditions is a fundamental aspect of calculus and differential equations. In this article, we will explore the detailed solution process for the initial value problem (IVP):

$$ Y'' - 2Y' - 3Y = 21t e^{2t} $$
with initial conditions:
$$ Y(0) = 4 \quad \text{and} \quad Y'(0) = 0. $$

This problem involves solving a nonhomogeneous second-order linear differential equation with constant coefficients. We will walk through each step comprehensively, including solving the homogeneous equation, finding a particular solution, applying initial conditions, and verifying the solution.

Understanding the Differential Equation

Type of Equation

The given differential equation:

$$ Y'' - 2Y' - 3Y = 21t e^{2t} $$

is a nonhomogeneous linear second-order differential equation with constant coefficients. The left side is the homogeneous part, and the right side is the nonhomogeneous part (forcing function).

Key Components

- Homogeneous part: $ Y'' - 2Y' - 3Y = 0 $
- Nonhomogeneous part (forcing term): $ 21t e^{2t} $
- Initial conditions: $ Y(0) = 4 $, $ Y'(0) = 0 $

Step 1: Solving the Homogeneous Equation

Finding the Characteristic Equation

The homogeneous differential equation:

$$Y'' - 2Y' - 3Y = 0$$

has characteristic equation:

$$r^2 - 2r - 3 = 0$$

Solving the Characteristic Equation

Factoring or using quadratic formula:

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-3)}}{2}$$
$$r = \frac{2 \pm \sqrt{4 + 12}}{2}$$
$$r = \frac{2 \pm \sqrt{16}}{2}$$
$$r = \frac{2 \pm 4}{2}$$

Thus, the roots are:

- $r_1 = \frac{2 + 4}{2} = 3$
- $r_2 = \frac{2 - 4}{2} = -1$

General Solution of the Homogeneous Equation

The homogeneous solution:

$$Y_h(t) = C_1 e^{3t} + C_2 e^{-t}$$

where C_1 and C_2 are arbitrary constants.

Step 2: Finding a Particular Solution

Understanding the Forcing Function

The right side is $21 t e^{2t}$. Since the forcing term involves $t e^{2t}$, we recognize this as a polynomial times an exponential function.

Method: Variation of Parameters or Undetermined Coefficients

Because the forcing function involves $t e^{2t}$, and e^{2t} is not part of the homogeneous solution, the method of undetermined coefficients is suitable.

However, since the forcing involves a polynomial times an exponential, the ansatz for the particular solution is:

$$Y_p(t) = e^{2t} (A t + B)$$

where A and B are constants to be determined.

Calculating $Y_p(t)$

1. Compute derivatives:

$$
Y_p(t) = e^{2t} (A t + B)
$$

$$
Y_p'(t) = \frac{d}{dt} [e^{2t} (A t + B)] = e^{2t} (A t + B) \times 2 + e^{2t} (A) = e^{2t} [2(A t + B) + A]
$$

$$
Y_p''(t) = \frac{d}{dt} [e^{2t} (2(A t + B) + A)]
$$

Applying the product rule:

$$
Y_p''(t) = e^{2t} \times 2 [2(A t + B) + A] + e^{2t} \times 2A
$$

Simplify:

$$
Y_p''(t) = e^{2t} [4(A t + B) + 2A + 2A] = e^{2t} [4A t + 4B + 4A]
$$

2. Substitute into the original differential equation:

$$
Y'' - 2Y' - 3Y = 21 t e^{2t}
$$

Plugging in the derivatives:

$$
e^{2t} [4A t + 4B + 4A] - 2 \times e^{2t} [2(A t + B) + A] - 3 \times e^{2t} (A t + B) = 21 t e^{2t}
$$

Factor out e^{2t}:

$$
e^{2t} \left[(4A t + 4B + 4A) - 2 (2A t + 2B + A) - 3 (A t + B) \right] = 21 t e^{2t}
$$

Simplify inside brackets:

$$
(4A t + 4B + 4A) - (4A t + 4B + 2A) - (3A t + 3B)
$$

Combine like terms:

- $4A t - 4A t = 0$
- $4B - 4B = 0$
- $4A - 2A = 2A$
- Remaining terms: $-3A t - 3B$

So, total:

$$
2A - 3A t - 3B
$$

Set equal to RHS:

$$
e^{2t} (2A - 3A t - 3B) = 21 t e^{2t}
$$

Divide both sides by e^{2t}:

$$
2A - 3A t - 3B = 21 t
$$

Now, equate coefficients of like powers of t:

- Coefficient of t:

$$
-3A = 21 \implies A = -7
$$

- Constant term:

$$
2A - 3B = 0 \implies 2(-7) - 3B = 0 \implies -14 - 3B = 0 \implies 3B = -14 \implies B = -\frac{14}{3}
$$

Particular Solution

Thus,

$$
Y_p(t) = e^{2t} \left(-7 t - \frac{14}{3} \right)
$$

Step 3: General Solution

The general solution to the differential equation is:

$$
Y(t) = Y_h(t) + Y_p(t)
$$

which gives:

$$
Y(t) = C_1 e^{3t} + C_2 e^{-t} + e^{2t} \left(-7 t - \frac{14}{3} \right)
$$

Step 4: Applying Initial Conditions

Initial Conditions

- $Y(0) = 4$
- $Y'(0) = 0$

Calculate $Y(0)$

At $t = 0$:

$$
Y(0) = C_1 e^{0} + C_2 e^{0} + e^{0} \left(-7 \times 0 - \frac{14}{3} \right) = C_1 + C_2 - \frac{14}{3}
$$

Set equal to 4:

$$
C_1 + C_2 - \frac{14}{3} = 4
$$

Multiply through by 3:

$$
3C_1 + 3C_2 - 14 = 12
$$

$$
3C_1 + 3C_2 = 26
$$

Divide by 3:

$$
C_1 + C_2 = \frac{26}{3}
$$

Calculate $Y'(t)$ and evaluate at $t=0$

Differentiate $Y(t)$:

$$
Y(t) = C_1 e^{3t} + C_2 e^{-t} + e^{2t} \left(-7 t - \frac{14}{3} \right)
$$

Derivative:

\[
Y'(t

Frequently Asked Questions

What is the initial value problem (IVP) given in the differential equation $Y'' - 2Y' - 3Y = 21te^{2t}$ with initial conditions $Y(0)=4$ and $Y'(0)=0$?

The IVP is a second-order linear differential equation $Y'' - 2Y' - 3Y = 21te^{2t}$ with initial conditions $Y(0)=4$ and $Y'(0)=0$.

How do you find the complementary solution for the differential equation $Y'' - 2Y' - 3Y = 0$?

Solve the characteristic equation $r^2 - 2r - 3 = 0$, which factors as $(r - 3)(r + 1)=0$, giving roots $r=3$ and $r=-1$. The complementary solution is $Y_c(t) = C_1 e^{3t} + C_2 e^{-t}$.

What method is used to find the particular solution for the nonhomogeneous differential equation $Y'' - 2Y' - 3Y = 21te^{2t}$?

The method of undetermined coefficients is used, assuming a particular solution of the form $Y_p(t) = e^{2t}(A t + B)$, since the right side involves te^{2t}.

How do you determine the coefficients A and B for the particular solution?

Substitute $Y_p(t) = e^{2t}(A t + B)$ into the differential equation, compute derivatives, and equate coefficients of like terms to solve for A and B.

After finding the general solution, how are the initial conditions used to find the constants C_1 and C_2?

Plug in $t=0$ into the general solution $Y(t)$ and its derivative $Y'(t)$, set them equal to the initial conditions $Y(0)=4$ and $Y'(0)=0$, and solve the resulting equations for C_1 and C_2.

What is the complete solution to the initial value problem after calculating all constants?

The complete solution is $Y(t) = C_1 e^{3t} + C_2 e^{-t} + e^{2t}(A t + B)$, with C_1, C_2, A, and B determined from initial conditions and the particular solution, providing the explicit function satisfying the IVP.

Additional Resources

Initial Value Problem (IVP) Solution Analysis: $Y'' - 2Y' - 3Y = 21t e^{2t}$, with $Y(0)=4$ and $Y'(0)=0$

Introduction

When tackling differential equations, especially initial value problems (IVPs), the goal is to find a function $Y(t)$ that satisfies both the differential equation and the given initial conditions. The problem at hand:

$$
Y'' - 2Y' - 3Y = 21 t e^{2t}
$$

with initial conditions:

$$
Y(0) = 4, \quad Y'(0) = 0
$$

is a classic second-order linear nonhomogeneous differential equation with constant coefficients, coupled with specific initial conditions. Understanding how to systematically approach such problems is crucial for students, engineers, and mathematicians alike.

This article provides an extensive, step-by-step exploration of solving this IVP, shedding light on the various techniques involved, including the homogeneous solution, particular solution, and applying initial conditions to find the unique solution.

Understanding the Equation Structure

Deconstructing the Differential Equation

The key to solving any linear differential equation is to recognize the structure:

$$
Y'' - 2Y' - 3Y = 21 t e^{2t}
$$

This is a second-order linear differential equation with constant coefficients, featuring a nonhomogeneous term $21 t e^{2t}$. The general solution will be a sum of:

- The complementary (homogeneous) solution, Y_h, derived from the associated homogeneous equation:

$$
Y'' - 2Y' - 3Y = 0
$$

- The particular solution, Y_p, which accounts for the nonhomogeneous (forcing) term $21 t e^{2t}$.

Step 1: Homogeneous Solution

Characteristic Equation

First, solve the homogeneous differential equation:

$$
Y'' - 2Y' - 3Y = 0
$$

The characteristic equation is:

$$
r^2 - 2r - 3 = 0
$$

Factoring, or using the quadratic formula:

$$
r = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-3)}}{2} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm \sqrt{16}}{2}
$$

$$
r = \frac{2 \pm 4}{2}
$$

This yields two real roots:

$$
r_1 = 3, \quad r_2 = -1
$$

Homogeneous Solution

Based on these roots, the homogeneous solution:

$$
Y_h(t) = C_1 e^{3t} + C_2 e^{-t}
$$

where C_1, C_2 are arbitrary constants to be determined later using initial conditions.

Step 2: Finding the Particular Solution Y_p

The right-hand side (forcing term):

$$
f(t) = 21 t e^{2t}
$$

suggests the use of the Method of Undetermined Coefficients or Variation of Parameters. Due to the form involving $t e^{2t}$, which is not a standard polynomial or exponential, Variation of Parameters is often more systematic here.

Method of Variation of Parameters

This method involves:

1. Using the homogeneous solutions $Y_1(t) = e^{3t}$ and $Y_2(t) = e^{-t}$.
2. Finding particular solutions in the form:

$$
Y_p(t) = u_1(t) Y_1(t) + u_2(t) Y_2(t)
$$

where $u_1(t), u_2(t)$ are functions to be determined.

Step 2.1: Computing the Wronskian

The Wronskian $W(t)$:

$$
W(t) = \begin{vmatrix}
Y_1 & Y_2 \\
Y_1' & Y_2'
\end{vmatrix} = e^{3t} \times (-e^{-t}) - (3e^{3t}) \times e^{-t}
$$

Calculating:

$$
W(t) = -e^{3t} e^{-t} - 3e^{3t} e^{-t} = -e^{2t} - 3e^{2t} = -4e^{2t}
$$

Step 2.2: Expressions for u_1' and u_2'

From Variation of Parameters theory:

$$
u_1' = - \frac{Y_2 \times f(t)}{W(t)} = - \frac{e^{-t} \times 21 t e^{2t}}{-4 e^{2t}}
$$

$$
u_2' = \frac{Y_1 \times f(t)}{W(t)} = \frac{e^{3t} \times 21 t e^{2t}}{-4 e^{2t}}
$$

Simplify each:

$$
u_1' = - \frac{e^{-t} \times 21 t e^{2t}}{-4 e^{2t}} = - \frac{21 t e^{(2t - t)}}{-4 e^{2t}} = - \frac{21 t e^{t}}{-4 e^{2t}}
$$

$$
u_2' = \frac{e^{3t} \times 21 t e^{2t}}{-4 e^{2t}} = \frac{21 t e^{(3t + 2t)}}{-4 e^{2t}} = \frac{21 t e^{5t}}{-4 e^{2t}}
$$

Further simplifications:

$$
u_1' = \frac{21 t e^{t}}{4 e^{2t}} = \frac{21 t}{4} e^{t - 2t} = \frac{21 t}{4} e^{-t}
$$

$$
u_2' = - \frac{21 t e^{5t}}{4 e^{2t}} = - \frac{21 t}{4} e^{5t - 2t} = - \frac{21 t}{4} e^{3t}
$$

Step 2.3: Integrate to find $u_1(t)$ and $u_2(t)$

Integrate u_1':

\[

u_1(t) = \int \frac{21 t}{4} e^{-t} dt
\]

Pull out the constant:

\[
u_1(t) = \frac{21}{4} \int t e^{-t} dt
\]

Use integration by parts:

Let:

- \(u = t \Rightarrow du = dt\)
- \(dv = e^{-t} dt \Rightarrow v = -e^{-t}\)

Then:

\[
\int t e^{-t} dt = - t e^{-t} + \int e^{-t} dt = - t e^{-t} - e^{-t} + C
\]

Hence:

\[
u_1(t) = \frac{21}{4} \left(- t e^{-t} - e^{-t}\right) + C_1
\]

Similarly, for \(u_2(t)\):

\[
u_2(t) = \int - \frac{21 t}{4} e^{3t} dt = - \frac{21}{4} \int t e^{3t} dt
\]

Apply integration by parts again:

- \(u = t \Rightarrow du = dt\)
- \(dv = e^{3t} dt \Rightarrow v = \frac{1}{3} e^{3t}\)

Then:

\[
\int t e^{3t} dt = \frac{1}{3} t e^{3t} - \frac{1}{3} \int e^{3t} dt = \frac{1}{3} t e^{3t} -
\frac{1}{3} \times \frac{1}{3} e^{3t} = \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t}
\]

Therefore:

\[
u_2(t) = - \frac{21}{4} \times \left(\frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t}\right) + C_2
\]

Simplify:

$$
u_2(t) = - \frac{21}{4} \times \frac{1}{3} t e^{3t} + \frac{21}{4} \times \frac{1}{9} e^{3t} + C_2 = - \frac{7}{4} t e^{3t} + \frac{7}{6} e^{3t}
$$
