

Fit A Quadratic Polynomial To The Data Points (0,27), (1,0), (2,0), (3,0), Using Least Squares. Sketch

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Introduction to Quadratic Polynomial Fitting

Fitting a quadratic polynomial to a set of data points is a fundamental task in data analysis and mathematical modeling. When data points are available, and the relationship between variables is believed to be quadratic in nature, the goal is to find a quadratic function that best approximates the data. This process is especially useful in various fields such as physics, engineering, economics, and biological sciences, where modeling the underlying trend of data points helps in predictions, understanding phenomena, and decision-making.

In this article, we will explore how to fit a quadratic polynomial to the specific data points (0, 27), (1, 0), (2, 0), and (3, 0) using the least squares method. We will also discuss how to sketch the resulting quadratic curve and interpret its significance.

Understanding the Data Points

The data points provided are:

- (0, 27)
- (1, 0)
- (2, 0)
- (3, 0)

Visually, these points suggest that the quadratic curve may have a peak or some specific shape. Observing the points, the value at $x=0$ is quite high (27), while at $x=1, 2$, and 3 , the y -values are zero. This pattern hints at a quadratic function that reaches a maximum somewhere near $x=0$ and then decreases to zero for larger x -values.

The Quadratic Polynomial Model

A quadratic polynomial generally takes the form:

$$y = ax^2 + bx + c$$

where:

- a , b , and c are real coefficients to be determined.

Our goal is to find the values of a , b , and c such that the polynomial best fits the given data points in the least squares sense.

Least Squares Method for Quadratic Fitting

The least squares method minimizes the sum of the squared residuals between the observed data points and the values predicted by the model. Given the data points (x_i, y_i) , the residual for each point is:

$$r_i = y_i - (ax_i^2 + bx_i + c)$$

The total error function S to minimize is:

$$S(a, b, c) = \sum_{i=1}^n r_i^2 = \sum_{i=1}^n [y_i - (ax_i^2 + bx_i + c)]^2$$

For our four data points, the sum becomes:

$$S(a, b, c) = (27 - c)^2 + (0 - (a + b + c))^2 + (0 - (4a + 2b + c))^2 + (0 - (9a + 3b + c))^2$$

Our task is to find a , b , and c that minimize S .

Formulating the Normal Equations

To minimize S , we take partial derivatives with respect to a , b , and c and set them to zero:

$$\begin{cases} \frac{\partial S}{\partial a} = 0, & \frac{\partial S}{\partial b} = 0, \\ \frac{\partial S}{\partial c} = 0 \end{cases}$$

This results in a system of three equations known as the normal equations:

$$\begin{cases} \sum x_i^4 \cdot a + \sum x_i^3 \cdot b + \sum x_i^2 \cdot c = \sum y_i x_i^2 \\ \sum x_i^3 \cdot a + \sum x_i^2 \cdot b + \sum x_i \cdot c = \sum y_i x_i \\ \sum x_i^2 \cdot a + \sum x_i \cdot b + n \cdot c = \sum y_i \end{cases}$$

where n is the number of data points.

Let's compute these sums explicitly.

Calculating Sums for the Data Points

x_i	y_i	x_i^2	x_i^3	x_i^4	$y_i x_i$	$y_i x_i^2$
0	27	0	0	0	0	0
1	0	1	1	1	0	0
2	0	4	8	16	0	0
3	0	9	27	81	0	0

Calculate the sums:

- $\sum x_i^2 = 0 + 1 + 4 + 9 = 14$
- $\sum x_i^3 = 0 + 1 + 8 + 27 = 36$
- $\sum x_i^4 = 0 + 1 + 16 + 81 = 98$
- $\sum y_i = 27 + 0 + 0 + 0 = 27$
- $\sum y_i x_i = 27 \times 0 + 0 \times 1 + 0 \times 2 + 0 \times 3 = 0$
- $\sum y_i x_i^2 = 27 \times 0 + 0 \times 1 + 0 \times 4 + 0 \times 9 = 0$

Now, the normal equations are:

$$\begin{cases} 98a + 36b + 14c = 0 \\ 36a + 14b + 4c = 0 \\ 14a + 4b + 4c = 27 \end{cases}$$

Solving the Normal Equations

Let's proceed to solve the system:

- Equation 1: $98a + 36b + 14c = 0$
- Equation 2: $36a + 14b + 4c = 0$
- Equation 3: $14a + 4b + 4c = 27$

Step 1: Express c from Equation 3:

$$14a + 4b + 4c = 27 \implies 4c = 27 - 14a - 4b \implies c = \frac{27 - 14a - 4b}{4}$$

Step 2: Substitute c into Equations 1 and 2.

- Substituting into Equation 2:

$$36a + 14b + 4 \times \frac{27 - 14a - 4b}{4} = 0$$

Simplify:

$$\begin{aligned} & \backslash[\\ & 36a + 14b + 27 - 14a - 4b = 0 \\ & \backslash] \end{aligned}$$

Combine like terms:

$$\begin{aligned} & \backslash[\\ & (36a - 14a) + (14b - 4b) + 27 = 0 \implies 22a + 10b + 27 = 0 \\ & \backslash] \end{aligned}$$

This simplifies to:

$$\begin{aligned} & \backslash[\\ & 22a + 10b = -27 \quad \text{\text{(Equation 4)}} \\ & \backslash] \end{aligned}$$

- Substituting (c) into Equation 1:

$$\begin{aligned} & \backslash[\\ & 98a + 36b + 14 \times \frac{27 - 14a - 4b}{4} = 0 \\ & \backslash] \end{aligned}$$

Simplify:

$$\begin{aligned} & \backslash[\\ & 98a + 36b + \frac{14 \times 27 - 14 \times 14a - 14 \times 4b}{4} = 0 \\ & \backslash] \end{aligned}$$

Calculate numerator:

$$\begin{aligned} & \backslash[\\ & 14 \times 27 = 378, \quad 14 \times 14a = 196a, \quad 14 \times 4b = 56b \\ & \backslash] \end{aligned}$$

So:

$$\begin{aligned} & \backslash[\\ & 98a + 36b + \frac{378 - 196a - 56b}{4} = 0 \\ & \backslash] \end{aligned}$$

Express as:

$$\begin{aligned} & \backslash[\\ & 98a + 36b + \frac{378}{4} - \frac{196a}{4} - \frac{56b}{4} = 0 \\ & \backslash] \end{aligned}$$

Simplify fractions:

$$\begin{aligned} & \backslash[\\ & 98a + 36b + 94.5 - 49a - 14b = 0 \\ & \backslash] \end{aligned}$$

Combine like terms:

$$\begin{aligned} & \backslash[\\ & (98a - 49a) + (36b - 14b) + 94.5 = 0 \implies \end{aligned}$$

Frequently Asked Questions

What is the goal of fitting a quadratic polynomial to the data points (0, 27), (1, 0), (2, 0), (3, 0) using least squares?

The goal is to find a quadratic polynomial $y = ax^2 + bx + c$ that best approximates the given data points by minimizing the sum of the squared differences between the observed and predicted values.

How do you set up the least squares problem for fitting a quadratic polynomial to the data points?

You set up a system of equations based on the data points, forming a design matrix with columns for x^2 , x , and 1, and then solve the normal equations to find the coefficients a , b , and c that minimize the sum of squared residuals.

What are the steps to compute the coefficients of the quadratic polynomial using least squares for these points?

First, construct the matrix X with rows $[x_i^2, x_i, 1]$, then set up the vector y with the corresponding y -values, and solve the normal equations $(X^T X)\beta = X^T y$ for $\beta = [a, b, c]$.

What is the significance of the data points (0,27), (1,0), (2,0), (3,0) in the context of polynomial fitting?

These points include one with a high y -value at $x=0$ and three points with $y=0$ at $x=1, 2$, and 3 , indicating the quadratic polynomial should pass close to these points, especially capturing the peak at $x=0$.

How can the least squares quadratic polynomial be sketched after computing its coefficients?

Plot the data points on a graph, then draw the quadratic curve defined by the fitted polynomial $y = ax^2 + bx + c$, ensuring the curve closely follows the data points, especially near the minimum at $x=1, 2, 3$.

What is the expected shape of the fitted quadratic polynomial for these data points?

The polynomial is expected to have a downward opening parabola with a maximum near $x=0$, matching the data points where $y=0$ at $x=1, 2, 3$, and peaking at $y=27$ when $x=0$.

Why is least squares fitting suitable for this data

set with multiple points on the x-axis where $y=0$?

Because least squares fitting finds the best approximation in a least-squares sense even when the data points do not lie perfectly on a quadratic curve, providing a smooth curve that best captures the overall trend.

Can the fitted quadratic polynomial exactly pass through all four data points in this case?

No, because there are more data points than polynomial coefficients; the least squares method finds the best fit that minimizes the overall error, so it generally won't pass through all points exactly unless the data is perfectly quadratic.

What tools or software can be used to perform the least squares fitting and sketching of the quadratic polynomial?

Tools such as MATLAB, Python (with NumPy and Matplotlib), R, or Excel can perform least squares regression and generate sketches of the fitted quadratic curve.

How do you interpret the coefficients of the fitted quadratic polynomial in the context of the data?

The coefficient a indicates the curvature (opening direction and steepness), b influences the slope, and c represents the y -intercept. Together, they describe the parabola that best fits the data in the least squares sense.

Additional Resources

Fit A Quadratic Polynomial To The Data Points $(0,27)$, $(1,0)$, $(2,0)$, $(3,0)$, Using Least Squares. Sketch

Fitting a quadratic polynomial to a set of data points using the least squares method is a fundamental technique in data analysis, numerical methods, and curve fitting. This approach aims to find the quadratic function $(y = ax^2 + bx + c)$ that best approximates the given data points by minimizing the sum of squared residuals—the differences between the observed and predicted values. In this article, we explore the process of fitting a quadratic polynomial to the points $(0,27)$, $(1,0)$, $(2,0)$, and $(3,0)$, discussing the mathematical formulation, solution steps, interpretation, and visualization.

Understanding the Problem and Data Points

The data points under consideration are:

- $(0, 27)$
- $(1, 0)$

- (2, 0)
- (3, 0)

These points suggest an initial pattern: the value of y drops from 27 at $x=0$ to zero at $x=1$, remains zero at $x=2$ and $x=3$. The goal is to find a quadratic polynomial $y = ax^2 + bx + c$ that best fits these points in the least squares sense.

The significance of choosing a quadratic polynomial stems from its flexibility to model curves with a single vertex and symmetry, making it suitable for data that exhibits a "bowl-shaped" or "parabolic" trend. Given that three points lie at $y=0$, the quadratic function might be expected to reflect this flat region with a suitable vertex, while still passing reasonably close to the initial high point at (0,27).

Mathematical Formulation of the Least Squares Problem

The quadratic polynomial has three unknown parameters: a , b , and c . The model is written as:

$$y = ax^2 + bx + c$$

Given the data points (x_i, y_i) , the residuals are:

$$r_i = y_i - (ax_i^2 + bx_i + c)$$

The least squares method minimizes the sum of squared residuals:

$$S(a, b, c) = \sum_{i=1}^n r_i^2 = \sum_{i=1}^n [y_i - (ax_i^2 + bx_i + c)]^2$$

For our four data points, the sum becomes:

$$S(a, b, c) = (27 - c)^2 + (0 - (a + b + c))^2 + (0 - (4a + 2b + c))^2 + (0 - (9a + 3b + c))^2$$

Our objective is to find the a , b , and c that minimize S .

Formulating the Normal Equations

Minimizing S involves solving for the parameters where partial

derivatives with respect to a , b , and c are zero:

$$\begin{aligned} \frac{\partial S}{\partial a} &= 0, \quad \frac{\partial S}{\partial b} = 0, \\ \frac{\partial S}{\partial c} &= 0 \end{aligned}$$

Expanding and simplifying these derivatives lead to a system of linear equations known as the normal equations:

$$\begin{cases} \sum x_i^4 \cdot a + \sum x_i^3 \cdot b + \sum x_i^2 \cdot c = \sum x_i^2 y_i \\ \sum x_i^3 \cdot a + \sum x_i^2 \cdot b + \sum x_i \cdot c = \sum x_i y_i \\ \sum x_i^2 \cdot a + \sum x_i \cdot b + n \cdot c = \sum y_i \end{cases}$$

Calculating the necessary sums:

$$\begin{aligned} - \sum x_i^4 &= 0^4 + 1^4 + 2^4 + 3^4 = 0 + 1 + 16 + 81 = 98 \\ - \sum x_i^3 &= 0 + 1 + 8 + 27 = 36 \\ - \sum x_i^2 &= 0 + 1 + 4 + 9 = 14 \\ - \sum x_i &= 0 + 1 + 2 + 3 = 6 \\ - \sum y_i &= 27 + 0 + 0 + 0 = 27 \\ - \sum x_i^2 y_i &= 0^2 \cdot 27 + 1^2 \cdot 0 + 2^2 \cdot 0 + 3^2 \cdot 0 = 0 + 0 + 0 + 0 = 0 \\ - \sum x_i y_i &= 0 \cdot 27 + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0 \end{aligned}$$

Plugging these into the normal equations:

$$\begin{aligned} 1. \quad 98a + 36b + 14c &= 0 \\ 2. \quad 36a + 14b + 6c &= 0 \\ 3. \quad 14a + 6b + 4c &= 27 \end{aligned}$$

Solving the System for Parameters

The system:

$$\begin{cases} 98a + 36b + 14c = 0 \\ 36a + 14b + 6c = 0 \\ 14a + 6b + 4c = 27 \end{cases}$$

We can solve this system step-by-step:

- From (2), express c :

$$6c = -36a - 14b \Rightarrow c = -6a - \frac{14}{6}b = -6a - \frac{7}{3}b$$

- Substitute (c) into (1):

$$98a + 36b + 14\left(-6a - \frac{7}{3}b\right) = 0$$

Simplify:

$$98a + 36b - 84a - \frac{98}{3}b = 0$$

$$(98a - 84a) + \left(36b - \frac{98}{3}b\right) = 0$$

$$14a + \left(36b - 32.\overline{6}b\right) = 0$$

$$14a + 3.\overline{3}b = 0$$

Express (a) :

$$14a = -3.\overline{3}b \rightarrow a = -\frac{3.\overline{3}}{14}b = -\frac{10/3}{14}b = -\frac{10}{3} \times \frac{1}{14}b = -\frac{10}{42}b = -\frac{5}{21}b$$

- Now, substitute (a) and (c) into (3):

$$14a + 6b + 4c = 27$$

Recall:

$$a = -\frac{5}{21}b$$

$$c = -6a - \frac{7}{3}b = -6\left(-\frac{5}{21}b\right) - \frac{7}{3}b = \frac{30}{21}b - \frac{7}{3}b = \frac{10}{7}b - \frac{7}{3}b$$

Express both over a common denominator, 21:

$$\frac{10}{7}b = \frac{30}{21}b$$

$$\frac{7}{3}b = \frac{49}{21}b$$

Thus,

$$c = \frac{30}{21}b - \frac{49}{21}b = -\frac{19}{21}b$$

Now substitute into (3):

$$14 \left(-\frac{5}{21}b \right) + 6b + 4 \left(-\frac{19}{21}b \right) = 27$$

Simplify:

$$-\frac{70}{21}b + 6b - \frac{76}{21}b = 27$$

Express all with denominator 21:

$$-\frac{70}{21}b + \frac{126}{21}b - \frac{76}{21}b = 27$$

Combine:

$$\frac{-70 + 126 - 76}{21}b = 27$$

$$(-70 + 126 - 76) = -20$$

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