If X1 And X2 Are Independent Nonnegative Continuous Random Variables, Show ThatP{X1 < X2| Min(X1,

If X1 And X2 Are Independent Nonnegative Continuous Random Variables, Show That P{X1 < X2 | Min(X1, X2)}

Understanding the behavior of random variables and their relationships is fundamental in probability theory and statistics. Specifically, analyzing the conditional probability involving two independent nonnegative continuous random variables, X1 and X2, provides insight into their comparative likelihoods. The problem asks us to demonstrate that, given the minimum of the two variables, the probability that X1 is less than X2 can be expressed in a particular form. This article thoroughly explores this problem, providing detailed explanations, step-by-step derivations, and relevant concepts necessary for a comprehensive understanding.

Introduction to the Problem

The statement involves two key random variables:

- X1 and X2: Independent, nonnegative, continuous random variables.

- Goal: To derive and understand the expression for the conditional probability \(P\{X_1 < X_2 \mid \min(X_1, X_2) \} \).

In probability, the notation indicates the probability that X1 is less than X2, given the value of their minimum. Since X1 and X2 are independent and nonnegative, their joint distribution has specific properties that facilitate this derivation.

Preliminaries and Definitions

Before delving into the derivation, it's essential to clarify some foundational concepts:

1. Independence of Random Variables

Two random variables X and Y are independent if their joint probability density function (pdf) factors into the product of their marginal pdfs:

$$
f_{X,Y}(x,y) = f_X(x) \times f_Y(y),
$$

for all x, y in their respective support.

2. Nonnegative Continuous Random Variables

X1 and X2 are nonnegative, meaning:

$$
P\{X_i \geq 0\} = 1, \quad i=1,2.
$$

Their pdfs are defined over [0, ∞):

$$
f_{X_i}(x), \quad x \geq 0.
$$

3. Minimum of Two Random Variables

$$
\min(X_1, X_2) = Z,
$$
which is itself a random variable. Its distribution depends on the distributions of X1 and X2.

Deriving the Conditional Probability

The main aim is to compute:

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\}.
$$

This involves understanding the joint distribution of X1 and X2, conditioned on their minimum being a specific value z.

Step 1: Express the Conditional Probability

By the definition of conditional probability:

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\} = \frac{P\{X_1 < X_2, \min(X_1, X_2) = z\}}{f_{\min}(z)},
$$

where $f_{\min}(z)$ is the pdf of $\min(X_1, X_2)$.

Since the minimum equals z, the joint behavior of X1 and X2 must satisfy:

$$
\min(X_1, X_2) = z,
$$
and both are ≥ z.

The numerator involves the joint probability that:

- $\min(X_1, X_2) = z$,
- and $X_1 < X_2$.

Step 2: Understanding the Event $\{X_1 < X_2, \min(X_1, X_2) = z\}$

For the minimum to be z, at least one of the variables must be equal to z, and the other must be greater than or equal to z.

The event $X_1 < X_2$ and $\min(X_1, X_2) = z$ can be partitioned into two disjoint cases:

- Case 1: $X_1 = z$ and $X_2 > z$,
- Case 2: $X_1 > z$ and $X_2 = z$.

Since in the first case, $X_1 = z$ and $X_2 > z$, and in the second, $X_2 = z$ and $X_1 > z$, for the event $X_1 < X_2$ to happen, only the first case contributes because:

- If $X_2 = z$ and $X_1 > z$, then $X_1 > X_2$, which is not consistent with $X_1 < X_2$.

Therefore, the relevant event is:

$$
\{X_1 = z, X_2 > z\}.
$$

And, similarly, for the denominator, the distribution of $\min(X_1, X_2)$, we need the joint pdf over the region where:

$$
X_1 = z, \quad X_2 \geq z,
$$

and vice versa, but since only the case where $X_1 = z$ and $X_2 > z$ contributes to the numerator, we focus on that.

Step 3: Expressing the Numerator

The probability that:

- $X_1 = z$,
- $X_2 > z$,
- and $X_1 < X_2$.

Given the independence and continuous nature, the probability that $X_1 = z$ is zero (since continuous variables have zero probability at a specific point). However, the joint probability density on the boundary is described via the joint pdf evaluated at the boundary.

To handle this, we consider the joint probability density functions over infinitesimal intervals:

$$
P\{X_1 \in [z, z + dz], X_2 > z\} \approx f_{X_1}(z) dz \times P\{X_2 > z\} = f_{X_1}(z) dz \times (1 - F_{X_2}(z)),
$$

where $F_{X_2}(z)$ is the CDF of X_2.

Similarly, for the joint distribution, the probability that $X_1 < X_2$, with X_1 near z and X_2 greater than z, is:

$$
\int_{x_2=z}^{\infty} f_{X_1}(z) \times f_{X_2}(x_2) dx_2,
$$

since independence implies the joint pdf factorizes.

Step 4: Formal Expression for the Conditional Probability

Putting it all together, the conditional probability becomes:

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\} = \frac{P\{X_1 = z, X_2 > z, X_1 < X_2\}}{f_{\min}(z)}.
$$

But as the probability of the variables taking exact values is zero, we use the density functions:

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\} = \frac{f_{X_1}(z) \int_{x_2=z}^{\infty} f_{X_2}(x_2) dx_2}{f_{\min}(z)}.
$$

Similarly, the pdf of the minimum, $f_{\min}(z)$, is known for independent variables:

$$
f_{\min}(z) = f_{X_1}(z) [1 - F_{X_2}(z)] + f_{X_2}(z) [1 - F_{X_1}(z)].
$$

Final Result and Interpretation

Using the above, the conditional probability simplifies to:

\[
\boxed{
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\} = \frac{f_{X_1}(z) [1 - F_{X_2}(z)]}{f_{X_1}(z) [1 - F_{X_2}(z)] + f_{X_2}(z) [1 - F_{X_1}(z)]}.
}
\]

This expression indicates that, given the minimum of the two variables is z, the probability that X1 is less than X2 depends on the relative densities and tail probabilities of the two distributions at z.

Special Cases and Intuition

- Identical Distributions: If \(X_1\) and \(X_2\) are identically distributed, then \(f_{X_1}(z) = f_{X_2}(z)\) and \(F_{X_1}(z) = F_{X_2}(z)\). The expression reduces to:

\[
P\{X_1 < X_2 \mid \min(X_1, X_2) = z\} = \frac{1 - F_{X}(

Frequently Asked Questions

What is the main goal when proving the independence of two nonnegative continuous random variables X1 and X2 in the context of the given problem?

The main goal is to show that the probability $P\{X1 < X2 | Min(X1, X2)\}$ equals the product of their individual probabilities, thereby establishing their independence conditioned on the minimum.

How does the independence of X1 and X2 simplify the calculation of conditional probabilities involving their order statistics?

Independence allows us to factor joint probabilities into the product of marginal probabilities, simplifying the analysis of events involving comparisons like $X1 < X2$ given the minimum.

What role does the joint distribution of (X1, X2) play in demonstrating

the conditional independence in this problem?

The joint distribution, which factors into the product of marginals due to independence, enables us to compute conditional probabilities explicitly and verify that conditioning on the minimum doesn't introduce dependence.

Why is it important that X1 and X2 are nonnegative in this proof?

Nonnegativity ensures the support of the variables is $[0, \infty)$, which simplifies the integration and probability calculations, especially when considering order statistics and minima.

Can the result be generalized if X1 and X2 are not independent? Why or why not?

No, the result relies heavily on the independence assumption. If X1 and X2 are dependent, their joint distribution cannot be factored into marginals, and the conditional independence of events like X1 < X2 given the minimum may not hold.

What is the significance of showing that P{X1 < X2 | Min(X1, X2)} = P{X1 < X2} in the context of statistical independence?

It demonstrates that the event X1 < X2 is unaffected by knowing the minimum, confirming the variables' independence extends to order statistics and related conditional events, which is fundamental in understanding their joint behavior.

Additional Resources

If X1 And X2 Are Independent Nonnegative Continuous Random Variables, Show That P{X1 < X2 | Min(X1, X2)}

Introduction

In the realm of probability theory, understanding the relationships between random variables is fundamental, especially when these variables are continuous and nonnegative—a common scenario in fields like reliability engineering, survival analysis, and queuing theory. A particularly interesting aspect arises when examining the probability that one variable is less than another, given the minimum of the two.

Suppose we have two independent, nonnegative, continuous random variables, X1 and X2. The question posed is: What is the probability that X1 is less than X2, conditioned on the minimum of the two? This problem not only tests our understanding of joint and conditional distributions but also offers insights into the behavior of independent variables under certain conditions.

In this article, we'll walk through the formal proof of the statement:

$$ P\{X_1 < X_2 \mid \min(X_1, X_2)\} $$

and explore the intuition behind the result, highlighting the key steps and underlying principles involved.

Understanding the Setting: Independent Nonnegative Continuous Random Variables

Before delving into the proof, it’s essential to clarify the assumptions and their implications:

- Independence: The variables X1 and X2 are independent, meaning the occurrence (or value) of one does not influence the other. Mathematically, their joint probability density function (pdf) factorizes as:

$$ f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2) $$

- Nonnegativity: Both variables take values in $[0, \infty)$. This simplifies certain integrals and ensures the minimum is well-defined over non-negative ranges.

- Continuity: The variables have continuous distributions, which guarantees that the probability of ties (i.e., $X_1 = X_2$) is zero, simplifying the analysis.

The Goal: Deriving the Conditional Probability

Our aim is to compute:

$$ P\{X_1 < X_2 \mid \min(X_1, X_2) = t\} $$

for a given value $t \geq 0$. The conditioned event is that the minimum of the two variables equals t, and within this condition, what is the probability that X_1 is less than X_2?

Step 1: Expressing the Conditional Probability

By the definition of conditional probability:

$$ P\{X_1 < X_2 \mid \min(X_1, X_2) = t\} = \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}} $$

Since both variables are continuous, the probability that the minimum equals a specific value t is zero; instead, we consider the distribution of the minimum and the joint events involving the variables conditioned on the minimum being close to t.

However, in continuous distributions, conditioning on $\{\min(X_1, X_2) = t\}$ involves considering the density of the minimum at t, and the joint distribution of the variables given that the minimum is t.

Step 2: Distribution of the Minimum

The distribution of the minimum of two independent nonnegative continuous variables is well-understood. The cumulative distribution function (CDF) of the minimum, $M = \min(X_1, X_2)$, is:

$$
F_M(t) = P(\min(X_1, X_2) \leq t) = 1 - P(X_1 > t, X_2 > t) = 1 - P(X_1 > t) P(X_2 > t)
$$

since independence holds. The associated probability density function (pdf) of the minimum is the derivative of $F_M(t)$:

$$
f_M(t) = \frac{d}{dt} F_M(t) = f_{X_1}(t) P(X_2 > t) + f_{X_2}(t) P(X_1 > t)
$$

This expression captures how the probability density of the minimum depends on the tail probabilities of the individual variables.

Step 3: Distribution of the Variables Given the Minimum

To analyze the probability that $X_1 < X_2$ given the minimum is t, we need the joint distribution of X_1 and X_2 conditioned on $\{\min(X_1, X_2) = t\}$.

Because the variables are continuous, the joint density conditioned on the minimum being t can be expressed via the joint density over the set:

$$
\{ (x_1, x_2) \mid \min(x_1, x_2) = t \}

\]

which geometrically corresponds to the union of the two lines:

- $x_1 = t, x_2 \geq t$,
- $x_2 = t, x_1 \geq t$.

The joint density conditioned on the minimum being t is proportional to the original joint density:

$$
f_{X_1, X_2}(x_1, x_2) \quad \text{for} \quad x_1, x_2 \geq t,
$$

with the constraint that either $x_1 = t$ or $x_2 = t$. Since the event $\{\min(X_1, X_2) = t\}$ has measure zero, we consider the conditional distribution of (X_1, X_2) given $\min(X_1, X_2) \in (t, t + dt)$, and then take limits as $dt \to 0$.

Step 4: Computing the Conditional Probability

The key insight is that, conditioned on the minimum being t, the remaining variable (the one larger than t) behaves as the original variable, shifted appropriately.

- If $X_1 = t$, then $X_2 \geq t$, with the distribution of X_2 conditioned on $X_2 \geq t$.
- Conversely, if $X_2 = t$, then $X_1 \geq t$, with the distribution of X_1 conditioned on $X_1 \geq t$.

Since the variables are independent, the joint conditioned distribution decomposes into a mixture:

$$
P\{\text{either } X_1 = t \text{ and } X_2 \geq t, \text{ or } X_2 = t \text{ and } X_1 \geq t\}
$$

with weights proportional to the respective probabilities.

Step 5: Symmetry and Final Result

Because of the symmetry in the problem and independence, the probability that $X_1 < X_2$ given the minimum is t turns out to be:

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = t\} = \frac{1}{2}
$$

This result stems from the fact that, conditioned on the minimum being t, the larger variable (either X_1 or X_2) is equally likely to be associated with either variable, and the independence ensures no bias.

Conclusion: The Key Result

Putting everything together, we arrive at the elegant and intuitive conclusion:

Given two independent nonnegative continuous random variables X_1 and X_2, the probability that X_1 is less than X_2 given their minimum is t is exactly $1/2$.

Mathematically,

$$
P\{X_1 < X_2 \mid \min(X_1, X_2) = t\} = \frac{1}{2}
$$

for all $t \geq 0$.

Implications and Intuition

This result aligns with the intuitive notion that, in the absence of any dependency or bias, the variables are "equally likely" to be smaller or larger than each other once conditioned on their minimum. It also underscores the symmetry inherent in independent, identically "shaped" distributions.

In practical applications, this insight provides a foundational understanding of the probabilistic structure when dealing with competing lifetimes, failure times, or service durations modeled by such variables.

Closing Remarks

The proof hinges on the independence and continuity of the variables, which eliminate complexities like ties or dependence biases. While the derivation involves detailed considerations of joint densities and conditioning, the final result is remarkably simple: the probability is always $1/2$, reflecting the inherent symmetry.

Understanding this fundamental principle enriches our comprehension of stochastic comparisons, and it exemplifies how elegant results often arise from seemingly complex probabilistic conditions. Whether in reliability analysis, risk assessment, or statistical modeling, recognizing such symmetry can guide both theoretical insights and practical decision-making.

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