In This Problem You Will Use Variation Of Parameters To Solve The Nonhomogeneous Equationy2y+y=4etA.

In This Problem You Will Use Variation Of Parameters To Solve The Nonhomogeneous Equation y'' + y = 4e^{t}A.

Understanding the Differential Equation y'' + y = 4e^{t}A

Differential equations are fundamental in modeling various physical phenomena, from heat transfer to population dynamics. The specific equation y'' + y = 4e^{t}A is a second-order linear nonhomogeneous differential equation. To solve this type of equation, especially when the nonhomogeneous part involves exponential functions, the method of variation of parameters is a powerful and versatile tool. This article guides you through understanding, applying, and mastering the variation of parameters technique for solving such equations.

What Is a Second-Order Linear Differential Equation?

A second-order linear differential equation has the general form:

$$ y'' + p(t) y' + q(t) y = g(t) $$

where:

- y is the unknown function of t,
- $p(t)$ and $q(t)$ are coefficient functions,
- $g(t)$ is the nonhomogeneous term (forcing function).

In our case, the equation simplifies to:

$$ y'' + y = 4e^{t}A $$

which is a special case with constant coefficients and no y' term.

Homogeneous vs. Nonhomogeneous Equations

Understanding the difference between homogeneous and nonhomogeneous equations is essential:

- Homogeneous Equation: $y'' + y = 0$

- Nonhomogeneous Equation: $y'' + y = 4e^{t}A$

The general solution to the nonhomogeneous equation is the sum of:

1. Complementary Solution (Solution to the homogeneous equation): y_c
2. Particular Solution (Specific to the nonhomogeneous part): y_p

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Solving the Homogeneous Equation y'' + y = 0

Step 1: Find the characteristic equation

$$r^2 + 1 = 0$$

Step 2: Solve for r

$$r = \pm i$$

Step 3: Write the homogeneous solution

$$y_c(t) = C_1 \cos t + C_2 \sin t$$

where C_1 and C_2 are arbitrary constants.

Applying Variation of Parameters to Find the Particular Solution

The key challenge lies in finding y_p, the particular solution, especially when the nonhomogeneous term involves exponential functions. The method of variation of parameters provides a systematic approach to this.

Why Use Variation of Parameters?

Unlike the method of undetermined coefficients, variation of parameters is applicable to a broader class of functions $g(t)$, including exponentials, polynomials, and their combinations.

The General Procedure

1. Determine the fundamental solutions $y_1(t)$ and $y_2(t)$ of the homogeneous equation.
2. Set up the particular solution as a linear combination with variable coefficients:

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

3. Derive equations for $u_1'(t)$ and $u_2'(t)$ based on the Wronskian.
4. Integrate to find $u_1(t)$ and $u_2(t)$.

5. Construct the particular solution $y_p(t)$.

Step-by-Step Solution Using Variation of Parameters

Step 1: Homogeneous solutions

From above,

$$ y_1(t) = \cos t $$
$$ y_2(t) = \sin t $$

Step 2: Wronskian calculation

The Wronskian $W(t)$ is:

$$ W(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t) $$

Calculating derivatives:

$$ y_1'(t) = -\sin t $$
$$ y_2'(t) = \cos t $$

Thus,

$$ W(t) = (\cos t)(\cos t) - (-\sin t)(\sin t) = \cos^2 t + \sin^2 t = 1 $$

Step 3: Find $u_1'(t)$ and $u_2'(t)$

Using the formulas:

$$ u_1'(t) = - \frac{ y_2(t) g(t) }{ W(t) } $$
$$ u_2'(t) = \frac{ y_1(t) g(t) }{ W(t) } $$

Given $g(t) = 4 e^{t} A$,

$$ u_1'(t) = - \sin t \times 4 e^{t} A $$
$$ u_2'(t) = \cos t \times 4 e^{t} A $$

Step 4: Integrate to find $u_1(t)$ and $u_2(t)$

$$
u_1(t) = -4A \int \sin t \, e^{t} dt
$$
$$
u_2(t) = 4A \int \cos t \, e^{t} dt
$$

These integrals involve the standard method of integration by parts.

Computing the Integrals

Integral for $u_1(t)$:

$$
I_1 = \int \sin t \, e^{t} dt
$$

Integration by parts:

Let:

- $u = \sin t \Rightarrow du = \cos t dt$
- $dv = e^{t} dt \Rightarrow v = e^{t}$

Applying integration by parts:

$$
I_1 = \sin t \, e^{t} - \int e^{t} \cos t dt
$$

The integral $\int e^{t} \cos t dt$ is similar; denote it as I_2.

Integral for $I_2 = \int e^{t} \cos t dt$:

- $u = \cos t \Rightarrow du = -\sin t dt$
- $dv = e^{t} dt \Rightarrow v = e^{t}$

Applying integration by parts:

$$
I_2 = \cos t \, e^{t} - \int e^{t} (-\sin t) dt = \cos t \, e^{t} + \int e^{t} \sin t dt
$$

But note $\int e^{t} \sin t dt = I_1$. Therefore:

$$
I_2 = \cos t \, e^{t} + I_1
$$

Recall I_1:

$$
I_1 = \sin t \, e^{t} - I_2
$$

Substitute I_2:

$$
I_1 = \sin t \, e^{t} - (\cos t \, e^{t} + I_1)
$$

$$
I_1 + I_1 = \sin t \, e^{t} - \cos t \, e^{t}
$$

$$
2 I_1 = e^{t} (\sin t - \cos t)
$$

$$
I_1 = \frac{1}{2} e^{t} (\sin t - \cos t)
$$

Now, compute $u_1(t)$:

$$
u_1(t) = -4A \times I_1 = -4A \times \frac{1}{2} e^{t} (\sin t - \cos t) = -2A e^{t} (\sin t - \cos t)
$$

Similarly, for $u_2(t)$:

$$
u_2(t) = 4A \int e^{t} \cos t dt = 4A \times I_2
$$

Recall:

$$
I_2 = \cos t \, e^{t} + I_1 = e^{t} \cos t + \frac{1}{2} e^{t} (\sin t - \cos t) = e^{t} \cos t + \frac{1}{2} e^{t} \sin t - \frac{1}{2} e^{t} \cos t
$$

Simplify:

$$
I_2 = \left(e^{t} \cos t - \frac{1}{2} e^{t} \cos t \right) + \frac{1}{2} e^{t} \sin t = \frac{1}{2} e^{t} \cos t + \frac{1}{2} e^{t} \sin t
$$

Therefore,

$$
u_2(t) = 4A \times I_2 = 4A \times \frac{1}{2} e^{t} (\cos t + \sin t) = 2A e^{t} (\cos t + \sin t)
$$

Forming the Particular Solution $y_p(t)$

Recall:

\[
$y_p(t) = u_1(t) y_1(t) +$

Frequently Asked Questions

What is the general approach to solving the differential equation $y'' + y = 4e^t$ using variation of parameters?

The approach involves first solving the homogeneous equation $y'' + y = 0$ to find the complementary solution, then using variation of parameters to find a particular solution for the nonhomogeneous equation by expressing it in terms of the homogeneous solutions and solving for the coefficients.

What are the homogeneous solutions for $y'' + y = 0$?

The homogeneous solutions are $y_h = C_1 \cos t + C_2 \sin t$, where C_1 and C_2 are arbitrary constants.

How do you set up the variation of parameters for this differential equation?

You assume a particular solution of the form $y_p = u_1(t) \cos t + u_2(t) \sin t$, where u_1 and u_2 are functions to be determined by solving a system derived from the original differential equation.

What is the formula for the derivatives of u_1 and u_2 in the variation of parameters method?

They are given by: $u_1' = - (y_2 g(t)) / W$, $u_2' = (y_1 g(t)) / W$, where $y_1 = \cos t$, $y_2 = \sin t$, $g(t) = 4e^t$, and W is the Wronskian of y_1 and y_2.

How do you compute the Wronskian W for the homogeneous solutions?

The Wronskian $W = y_1 y_2' - y_1' y_2$. For $y_1 = \cos t$ and $y_2 = \sin t$, $W = \cos t \cos t - (-\sin t) \sin t = \cos^2 t + \sin^2 t = 1$.

What is the integral expression for $u_1(t)$ and $u_2(t)$ in this problem?

$u_1(t) = -\int (y_2 g(t)) / W \, dt = -\int (\sin t \, 4e^t) \, dt$ and $u_2(t) = \int (y_1 g(t)) / W \, dt = \int (\cos t \, 4e^t) \, dt$.

How do you evaluate the integrals for $u_1(t)$ and $u_2(t)$?

These integrals involve integrating expressions like $4e^t \sin t$ and $4e^t \cos t$, which can be done using integration by parts or recognizing they are standard integrals involving exponential and trigonometric functions.

What is the final form of the particular solution $y_p(t)$ after applying variation of parameters?

After computing $u_1(t)$ and $u_2(t)$, substitute back into $y_p = u_1(t) \cos t + u_2(t) \sin t$, resulting in an explicit expression involving integrals of e^t multiplied by sine and cosine functions, which can be evaluated to obtain the particular solution.

Additional Resources

In This Problem You Will Use Variation Of Parameters To Solve The Nonhomogeneous Equation $y'' + y = 4e^t$

Introduction

In the realm of differential equations, solving nonhomogeneous linear differential equations stands as a fundamental challenge that often requires sophisticated methods beyond straightforward integration. The problem at hand—solving the differential equation $y'' + y = 4e^t$—presents an elegant opportunity to explore the variation of parameters technique, a powerful and versatile method for finding particular solutions when the method of undetermined coefficients is unsuitable or unwieldy.

This article aims to provide a comprehensive, detailed, and analytical exploration of how to employ variation of parameters to solve this specific second-order linear differential equation. We will dissect each step meticulously, elucidate the underlying concepts, and analyze the significance of the solution process within the broader context of differential equations. By the end, readers should have a clear understanding of the method's application, its advantages, and its place in the toolbox of differential equations.

Understanding the Differential Equation

The Structure of the Equation

The given differential equation is:

$$
y'' + y = 4e^t
$$

This is a second-order linear nonhomogeneous differential equation with constant coefficients. The left side, $y'' + y$, represents the homogeneous part, while the right side, $4e^t$, is the nonhomogeneous (forcing) term. The general solution to this type of equation comprises two parts:

$$
y(t) = y_h(t) + y_p(t)
$$

where:

- $y_h(t)$ is the homogeneous solution, solving $y'' + y = 0$,
- $y_p(t)$ is the particular solution, accounting for the nonhomogeneous term.

Significance of the Method

While the homogeneous solution can be found straightforwardly using characteristic equations, the particular solution often demands more nuanced techniques, especially when the nonhomogeneous term does not match the form suitable for undetermined coefficients. The variation of parameters method offers a systematic approach applicable in these contexts, making it a crucial technique in the differential equations toolkit.

Step 1: Solving the Homogeneous Equation

Deriving the Homogeneous Solution

The homogeneous equation:

$$
y'' + y = 0
$$

has characteristic equation:

$$
r^2 + 1 = 0
$$

which yields complex roots:

$$
r = \pm i

\]

Therefore, the homogeneous solution is:

$$
\boxed{
y_h(t) = C_1 \cos t + C_2 \sin t
}
$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Step 2: Setting Up the Variation of Parameters

Why Use Variation of Parameters?

When the nonhomogeneous term $4e^t$ does not fit the form for the method of undetermined coefficients (which typically works well for polynomial, exponential, sine, or cosine functions, or their combinations), variation of parameters provides a systematic alternative.

Fundamental Solution Set

The method begins with the fundamental set of solutions from the homogeneous equation:

$$
\begin{cases}
y_1(t) = \cos t \\
y_2(t) = \sin t
\end{cases}
$$

The goal is to find a particular solution of the form:

$$
y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)
$$

where $u_1(t)$ and $u_2(t)$ are functions to be determined, rather than constants.

Step 3: Formulating the Variation of Parameters

Deriving the Formulas for $u_1(t)$ and $u_2(t)$

The method involves solving for $u_1(t)$ and $u_2(t)$ using the system:

$$
\begin{cases}
u_1' y_1 + u_2' y_2 = 0 \\
u_1' y_1' + u_2' y_2' = g(t)
\end{cases}
$$

where $g(t) = 4e^t$ in our case.

Calculating the Wronskian:

$$
W = y_1 y_2' - y_2 y_1' = \cos t \cdot \cos t - \sin t \cdot (-\sin t) = \cos^2 t + \sin^2 t = 1
$$

The formulas for u_1' and u_2' are:

$$
u_1' = - \frac{y_2 \cdot g(t)}{W} = -\sin t \cdot 4e^t
$$

$$
u_2' = \frac{y_1 \cdot g(t)}{W} = \cos t \cdot 4e^t
$$

Step 4: Integrating to Find $u_1(t)$ and $u_2(t)$

Computing $u_1(t)$

$$
u_1(t) = - \int \sin t \cdot 4 e^t \, dt
$$

Similarly, for $u_2(t)$:

$$
u_2(t) = \int \cos t \cdot 4 e^t \, dt
$$

These integrals involve products of exponential and trigonometric functions, which are classic candidates for integration via integration by parts or recognizing standard integral forms.

Step 5: Evaluating the Integrals

Integral for $u_1(t)$:

$$
u_1(t) = -4 \int e^t \sin t \, dt
$$

Applying integration by parts:

Let:

- $u = \sin t$, so $du = \cos t \, dt$,
- $dv = e^t dt$, so $v = e^t$.

Then:

$$
\int e^t \sin t \, dt = e^t \sin t - \int e^t \cos t \, dt
$$

Now, compute:

$$
I = \int e^t \cos t \, dt
$$

Again, use integration by parts:

- $u = \cos t$, $du = - \sin t \, dt$,
- $dv = e^t dt$, $v = e^t$.

So:

$$
I = e^t \cos t + \int e^t \sin t \, dt
$$

Note that the integral $\int e^t \sin t \, dt$ appears on both sides. Denote:

$$
J = \int e^t \sin t \, dt
$$

From earlier:

$$
J = e^t \sin t - I
$$

and from the second integral:

$$
I = e^t \cos t + J
$$

Rearranged:

$$
J = e^t \sin t - I
$$

$$
I = e^t \cos t + J
$$

Substitute J into the second:

$$
I = e^t \cos t + e^t \sin t - I
$$

which simplifies to:

$$
2I = e^t (\sin t + \cos t)
$$

$$
I = \frac{e^t (\sin t + \cos t)}{2}
$$

Now, recall:

$$
J = e^t \sin t - I = e^t \sin t - \frac{e^t (\sin t + \cos t)}{2} = \frac{2 e^t \sin t - e^t (\sin t + \cos t)}{2} = \frac{e^t (\sin t - \cos t)}{2}
$$

Finally, the integral for $u_1(t)$:

$$
u_1(t) = -4 J = -4 \cdot \frac{e^t (\sin t - \cos t)}{2} = -2 e^t (\sin t - \cos t)
$$

Integral for $u_2(t)$:

$$
u_2(t) = 4 \int e^t \cos t \, dt
$$

Similarly, applying integration by parts:

\[
K = \int e^t \cos t \, dt
\]

Using previous results:

\[
K = e^t \cos t + \int e^t \sin t \, dt
\]

From earlier, \(J = \int e^t \sin t \, dt = \frac{e^t (\sin t - \cos t)}{2} \).

Thus:

\[
K = e^t \cos t + J = e^t \cos t + \frac{e^t (\sin t - \cos t)}{2} = \frac{2 e^t \cos t + e^t (\sin t - \cos t)}{2} = \frac{e^t (\sin t + \cos t)}{2}
\]

Therefore:

\[
u_2(t) =

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