Integrate The Function (x2+y2)12over The Region E That Is Bounded By The Xy Plane Below And Above By

Integrate The Function $(x^2 + y^2)^{12}$ over The Region E That Is Bounded By The XY Plane Below And Above By

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Introduction

In the realm of multivariable calculus, evaluating integrals over specific regions in space is fundamental for understanding physical phenomena, such as mass, charge distribution, and fluid flow. One common type of integral involves integrating functions over regions bounded by surfaces and planes, often requiring the use of advanced coordinate systems like cylindrical or spherical coordinates to simplify calculations.

This article focuses on a particular problem: integrating the function $(x^2 + y^2)^{12}$ over a region E that is bounded below by the xy-plane and above by a certain surface. This problem exemplifies the application of double and triple integrals, coordinate transformations, and symmetry considerations in calculus.

By exploring this integral in detail, you will learn how to set up the integral, choose appropriate coordinate systems, and evaluate the integral step-by-step. Whether you're a student preparing for calculus exams or a professional dealing with mathematical modeling, understanding this process is essential for mastering multivariable calculus.

Understanding the Region E

The Geometric Boundary of the Region E

The region E in three-dimensional space is described as being bounded below by the xy-plane and above by a certain surface. Typically, such a surface might be expressed in terms of z, like z = f(x, y).

Key points include:

- Lower boundary: The xy-plane, where z = 0.
- Upper boundary: A surface, which could be expressed as z = g(x, y) or in other forms depending on the problem specifics.

Assuming the problem states "bounded above by the surface z = h(x, y)," the region E is:

$$
E = \{(x, y, z) \mid (x, y) \in D, \quad 0 \leq z \leq h(x, y)\}
$$

where D is the projection of E onto the xy-plane:

$$
D = \{(x, y) \mid (x, y, 0) \in E\}
$$

The Function to Integrate

The function is:

$$
f(x, y) = (x^2 + y^2)^{12}
$$

Note that this function depends only on x and y, which simplifies integration, especially when employing cylindrical coordinates.

Choosing the Appropriate Coordinate System

Why Use Cylindrical Coordinates?

The integrand $(x^2 + y^2)^{12}$ naturally suggests the use of cylindrical coordinates because:

- The expression $x^2 + y^2$ converts directly to r^2.
- The symmetry about the z-axis simplifies integration over circular or radial regions.

Cylindrical Coordinates Overview

The transformation from Cartesian to cylindrical coordinates is:

$$
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
$$

The differential volume element becomes:

$$
dV = r \, dr \, d\theta \, dz
$$

For integrating a function $f(x, y)$ over a region E, the integral becomes:

$$
\iiint_E f(x, y) \, dV
$$

which, in cylindrical coordinates, simplifies to:

$$
\int_{\theta = \theta_1}^{\theta_2} \int_{r = r_1}^{r_2} \int_{z=0}^{h(r, \theta)} f(r, \theta) \, r \, dz \, dr \, d\theta
$$

Setting Up the Integral

Step 1: Describe the Projection Region D in the xy-Plane

Depending on the surface that bounds E from above, the projection D in the xy-plane could be a circle or another shape.

Common scenarios include:

- Full circle: r ranges from 0 to some radius R, and θ from 0 to 2π.
- Partial circle or sector: r over a subset of the radius, θ over some interval.

For the sake of this example, assume the upper boundary is a sphere or cylinder that creates a circular region in the xy-plane.

Step 2: Express the Upper Surface in Cylindrical Coordinates

Suppose the upper surface is a sphere of radius R, centered at the origin:

$$
z = \sqrt{R^2 - r^2}
$$

Then, the region E is the volume beneath the sphere and above the xy-plane:

$$
0 \leq z \leq \sqrt{R^2 - r^2}
$$

and

$$
0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi
$$

Step 3: Write the Integral

The triple integral becomes:

$$
\iiint_E (x^2 + y^2)^{12} \, dV = \int_{0}^{2\pi} \int_{0}^{R} \int_{0}^{\sqrt{R^2 - r^2}} (r^2)^{12} \, r \, dz \, dr \, d\theta
$$

Note that:

$$
f(r, \theta) = (r^2)^{12} = r^{24}
$$

and the volume element:

$$
dV = r \, dz \, dr \, d\theta
$$

Step-by-step Evaluation of the Integral

Step 1: Integrate with respect to z

$$
\int_{0}^{\sqrt{R^2 - r^2}} dz = \sqrt{R^2 - r^2}
$$

Step 2: Integrate with respect to r

The integral reduces to:

\[

$$
\int_{0}^{2\pi} d\theta \int_{0}^{R} r^{24} \sqrt{R^2 - r^2} \, dr
$$

The θ integral simplifies to:

$$
2\pi
$$

So, the problem is:

$$
2\pi \int_{0}^{R} r^{24} \sqrt{R^2 - r^2} \, dr
$$

Step 3: Use substitution for the radial integral

Let:

$$
r = R \sin \phi \quad \Rightarrow \quad dr = R \cos \phi \, d\phi
$$

When:

- $r = 0 \Rightarrow \phi = 0$
- $r = R \Rightarrow \phi = \pi/2$

Substituting:

$$
r^{24} = R^{24} \sin^{24} \phi
$$
$$
\sqrt{R^2 - r^2} = R \cos \phi
$$
$$
dr = R \cos \phi \, d\phi
$$

The integral becomes:

$$
2\pi R^{24} \int_{0}^{\pi/2} \sin^{24} \phi \times R \cos \phi \times R \cos \phi \, d\phi
$$

which simplifies to:

$$
2\pi R^{26} \int_{0}^{\pi/2} \sin^{24} \phi \times \cos^{2} \phi \, d\phi
$$

Final Expression and Evaluation

The integral reduces to:

\[

$$
I = 2\pi R^{26} \int_{0}^{\pi/2} \sin^{24} \phi \times \cos^{2} \phi \, d\phi
$$

This is a standard integral involving powers of sine and cosine, which can be evaluated using Beta and Gamma functions.

Using Beta and Gamma Functions

Recall:

$$
\int_{0}^{\pi/2} \sin^{m} \phi \, \cos^{n} \phi \, d\phi = \frac{1}{2} \mathrm{B}\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n}{2} + 1\right)}
$$

In our case, $m = 24$, $n=2$:

$$
\int_{0}^{\pi/2} \sin^{24} \phi \times \cos^{2} \phi \, d\phi = \frac{1}{2} \frac{\Gamma\left(\frac{25}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{24+2}{2} + 1\right)} = \frac{1}{2} \frac{\Gamma\left(12.5\right) \Gamma\left(1.5\right)}{\Gamma(14)}
$$

Step 4: Express the Gamma functions

- \(\Gamma(1.5) = \

Frequently Asked Questions

What is the main goal when integrating the function $(x^2 + y^2)^{12}$ over the region E?

The main goal is to find the total value of the function over the specified region E, which involves computing a double integral that accounts for the shape and bounds of the region in the xy-plane.

How do I determine the bounds of integration for the region E bounded below by the xy-plane?

Since the region is bounded below by the xy-plane, the lower limit for y is typically 0, and the bounds for x depend on the shape of the region above the xy-plane. The specific bounds need to be identified based on the description of the upper boundary.

What coordinate system is most suitable for integrating $(x^2 + y^2)^{12}$ over the region E?

Using polar coordinates is most suitable because the integrand involves $x^2 + y^2$, which simplifies to r^2 in polar coordinates, making the integration more straightforward.

How do I convert the integrand $(x^2 + y^2)^{12}$ into polar coordinates?

In polar coordinates, $x = r \cos \theta$ and $y = r \sin \theta$, so $x^2 + y^2 = r^2$. Therefore, $(x^2 + y^2)^{12}$ becomes $(r^2)^{12} = r^{24}$.

What is the general form of the double integral in polar coordinates for this problem?

The double integral over region E in polar coordinates becomes ⍰ ⍰ ⍰ r from r_{min} to r_{max} and heta from heta_{min} to heta_{max} of r^{24} r dr d heta, simplifying to ⍰ ⍰ ⍰ r^{25} dr d heta.

How do I determine the limits for r and θ in the polar coordinate setup?

The limits for r depend on the boundary of the region above the xy-plane, often from 0 to a maximum radius determined by the upper boundary. The limits for θ typically range from 0 to 2π if the region is circularly symmetric, or adjusted accordingly if the region is bounded differently.

What is the significance of symmetry in simplifying the integration process for this problem?

Symmetry can reduce the complexity of the integral, especially if the region and the integrand are symmetric about an axis or origin, allowing you to evaluate the integral over a portion and multiply accordingly, or simplifying the bounds and integrand.

Additional Resources

Integrate the Function $(x^2 + y^2)^{12}$ over the Region E Bounded by the XY Plane and an Upper Surface

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Introduction

In the realm of multivariable calculus, the task of integrating a function over a specific region in the xy-plane or in three-dimensional space is foundational for understanding physical phenomena, geometric properties, and mathematical modeling. One such problem involves evaluating the integral of the function $(x^2 + y^2)^{12}$ over a region designated as E, which is bounded below by the xy-plane and above by some surface. This problem combines geometric intuition with advanced calculus techniques, notably the use of coordinate transformations such as polar or cylindrical coordinates.

This article aims to provide a comprehensive, detailed, and analytical exploration of this problem. We will establish the nature of the region E, interpret the integrand, choose appropriate coordinate systems, and work through the integral's evaluation systematically. The discussion will also extend to the implications of the integral—such as physical interpretations, properties, and potential applications—while emphasizing the mathematical principles underlying the solution.

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1. Understanding the Region E

1.1. Geometric Description of Region E

The problem states that E is bounded below by the xy-plane and above by some surface. To analyze this region precisely, we need to clarify the surface that caps E. Typically, in such problems, the upper surface might be a paraboloid, sphere, or some other surface defined explicitly or implicitly.

Suppose the upper boundary is given by:

$$ z = f(x, y) $$

where $f(x, y)$ is a surface function.

For the purpose of this discussion, let's assume the upper surface is a paraboloid:

$$ z = R^2 - (x^2 + y^2) $$

which opens downward and intersects the xy-plane at the boundary where $z=0$:

$$ R^2 - (x^2 + y^2) = 0 \Rightarrow x^2 + y^2 = R^2 $$

Thus, the region E is a paraboloid cap of height R^2, bounded below by the xy-plane ($z=0$) and above by the paraboloid surface. The projection of E onto the xy-plane is therefore the disk:

$$ D = \{ (x, y) \mid x^2 + y^2 \leq R^2 \} $$

This scenario is common in volume and surface integrals, especially in problems involving rotational symmetry.

1.2. General Boundaries and Assumptions

- Lower boundary: xy-plane ($z=0$)
- Upper boundary: $z = R^2 - (x^2 + y^2)$

Note: If the problem differs—for example, if the upper boundary is a different surface—the analysis must adapt accordingly. The key is to identify the bounds in the xy-plane and the surface function to set up the integral correctly.

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2. The Integrand: $(x^2 + y^2)^{12}$

2.1. Nature of the Function

The integrand, $(x^2 + y^2)^{12}$, is a radially symmetric function centered at the origin. Its value depends solely on the distance from the origin, as:

$$ r = \sqrt{x^2 + y^2} $$

and

$$ (x^2 + y^2)^{12} = r^{24} $$

This symmetry simplifies the integral significantly when switching to polar or cylindrical coordinates, as the integrand becomes a function only of r.

2.2. Implications of Symmetry

Radial symmetry implies the integral over a circular region simplifies because the integrand is invariant under rotations. This symmetry is exploited in the change of variables from Cartesian to polar coordinates, leading to a more straightforward evaluation.

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3. Choosing the Coordinate System

3.1. Cartesian vs. Polar Coordinates

Given the symmetry and the shape of the region, switching to polar coordinates (r, θ) is advantageous. The relationships are:

$$
x = r \cos \theta, \quad y = r \sin \theta, \quad dx\,dy = r\,dr\,d\theta
$$

The limits for r and θ depend on the projection of E onto the xy-plane.

3.2. The Limits in Polar Coordinates

- Angular coordinate θ: Since the projection is a disk, θ spans from 0 to 2π:

$$
0 \leq \theta \leq 2\pi
$$

- Radial coordinate r: Ranges from 0 to the boundary radius R:

$$
0 \leq r \leq R
$$

- Vertical coordinate z: Ranges from 0 (xy-plane) up to the surface:

$$
0 \leq z \leq R^2 - r^2
$$

---

4. Setting Up the Integral

4.1. Volume Element and Integration

The volume integral of the function over the region E is expressed as:

$$
V = \iiint_{E} (x^2 + y^2)^{12} \, dV
$$

Given the bounds:

- z varies between 0 and $f(r) = R^2 - r^2$,
- (r, θ) cover the disk $r \in [0, R], \theta \in [0, 2\pi]$.

The triple integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{R} \int_{0}^{R^2 - r^2} r^{24} \, dz \, r\, dr \, d\theta
$$

Note that the integrand includes $(x^2 + y^2)^{12} = r^{24}$, and the volume element $dV = dz\, dx\, dy = dz\, r\, dr\, d\theta$.

4.2. Performing the z-Integration

Since the integrand does not depend on z, the integration over z is straightforward:

$$
\int_{0}^{R^2 - r^2} dz = R^2 - r^2
$$

Thus, the integral simplifies to:

$$
V = \int_{0}^{2\pi} \int_{0}^{R} r^{24} (R^2 - r^2) \, r\, dr \, d\theta
$$

or,

$$
V = 2\pi \int_{0}^{R} r^{25} (R^2 - r^2) \, dr
$$

---

5. Evaluation of the Integral

5.1. The Radial Integral

The integral reduces to:

$$
I = \int_{0}^{R} r^{25} (R^2 - r^2) \, dr
$$

which can be expanded as:

$$
I = R^2 \int_{0}^{R} r^{25} \, dr - \int_{0}^{R} r^{27} \, dr
$$

Calculating each integral separately:

$$
\int_{0}^{R} r^{n} dr = \frac{R^{n+1}}{n+1}
$$

So,

$$
I = R^2 \left(\frac{R^{26}}{26} \right) - \left(\frac{R^{28}}{28} \right) = \frac{R^{28}}{26} - \frac{R^{28}}{28}
$$

Simplify:

$$
I = R^{28} \left(\frac{1}{26} - \frac{1}{28} \right) = R^{28} \left(\frac{28 - 26}{26 \times 28} \right) = R^{28} \left(\frac{2}{728} \right)
$$

Reduce the fraction:

$$
\frac{2}{728} = \frac{1}{364}
$$

Thus,

$$
I = \frac{R^{28}}{364}
$$

5.2. Final Volume Calculation

Recall the angular integral:

$$
V = 2\pi \times I = 2\pi \times \frac{R^{28}}{364} = \frac{2 \pi R^{28}}{364}
$$

Simplify numerator and denominator:

$$
V = \frac{\pi R^{28}}{182}
$$

---

6. Physical and Mathematical Implications

6.1. Interpretation of the Result

The integral computes a kind of "weighted volume" where the weight is $(x^2 + y^2)^{12}$. In physical contexts, this could represent a mass distribution where density increases with the radial distance from the origin.

The result:

$$
V = \frac{\pi R^{28}}{182}
$$

demonstrates a rapid growth with the radius R, reflecting the high exponent in the integrand.

6.2. Symmetry

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