

Let $B = \{b_1, b_2\}$ And $C = \{c_1, c_2\}$ Be Bases For $\mathbb{R}^2$. Find The Change-of-coordinates Matrix From B To C

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Introduction

In linear algebra, understanding how to switch between different coordinate systems or bases is fundamental. When working with vector spaces such as $\mathbb{R}^2$, the ability to convert coordinate representations from one basis to another simplifies many computations, including solving systems, transforming vectors, and analyzing linear transformations.

This article explores the process of finding the change-of-coordinates matrix from one basis to another in $\mathbb{R}^2$. Specifically, given two bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for $\mathbb{R}^2$, we will determine the matrix that transforms coordinates of vectors expressed in basis B into their coordinates in basis C. This process involves understanding how to express basis vectors of B in terms of C, assembling these expressions into a matrix, and using this matrix to perform change-of-coordinates.

Understanding Bases in $\mathbb{R}^2$

What Is a Basis?

A basis for $\mathbb{R}^2$ is a set of two vectors that are linearly independent and span the entire space. Every vector in $\mathbb{R}^2$ can be expressed uniquely as a linear combination of the basis vectors.

The Role of Change-of-coordinates Matrices

Given two bases, B and C , for $\mathbb{R}^2$, the change-of-coordinates matrix from B to C allows us to convert coordinate vectors from the B -coordinates to C -coordinates. Formally, if a vector v has coordinates $[v]_B$ in basis B , then its coordinates in basis C , $[v]_C$, are obtained via:

$$[v]_C = P_{\{C \leftarrow B\}} \times [v]_B$$

where $P_{\{C \leftarrow B\}}$ is the change-of-coordinates matrix from basis B to basis C .

Theoretical Foundations

Expressing Basis Vectors

To find $P_{\{C \leftarrow B\}}$, we need to express each basis vector of B in terms of the basis C :

$$b_i = \alpha_{i1} c_1 + \alpha_{i2} c_2$$

for $(i = 1, 2)$.

Constructing the Change-of-coordinates Matrix

Once we have expressed each b_i in terms of C , the coefficients form the columns of the matrix $P_{\{C \leftarrow B\}}$:

$$P_{C \rightarrow B} = [[b_1]_C \quad [b_2]_C]$$

where each column is the coordinate vector of (b_i) relative to C .

Step-by-Step Procedure to Find the Change-of-coordinates Matrix

Step 1: Identify the Basis Vectors

Suppose the bases are explicitly given as:

$$B = \{b_1, b_2\} \quad \text{and} \quad C = \{c_1, c_2\}$$

with:

$$b_1 = \begin{bmatrix} x_{b1} \\ y_{b1} \end{bmatrix}, \quad b_2 = \begin{bmatrix} x_{b2} \\ y_{b2} \end{bmatrix}$$

and

$$c_1 = \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix}, \quad c_2 = \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix}$$

Step 2: Express b_1 and b_2 in Terms of c_1 and c_2

Solve for scalars a_{11}, a_{12} such that:

$$b_1 = a_{11} c_1 + a_{12} c_2$$

Similarly, find a_{21}, a_{22} for b_2 :

$$b_2 = a_{21} c_1 + a_{22} c_2$$

This involves solving a system of linear equations:

$$\begin{cases} x_{b1} = a_{11} x_{c1} + a_{12} x_{c2} \\ y_{b1} = a_{11} y_{c1} + a_{12} y_{c2} \end{cases}$$

and similarly for b_2 .

Step 3: Form the Change-of-coordinates Matrix

Once the coefficients are found, the columns of the change-of-coordinates matrix $P_{C \leftarrow B}$ are:

$$P_{C \leftarrow B} =$$

$P_{C \leftarrow B} = \begin{bmatrix}$

$a_{11} \quad a_{21} \quad \backslash \quad$

$a_{12} \quad a_{22}$

$\end{bmatrix}$

$\backslash$

This matrix, when multiplied by a coordinate vector in B, yields the coordinate vector in C.

Practical Example: Computing the Change-of-coordinates Matrix

Suppose:

$\backslash$

$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\backslash$

and

$\backslash$

$C = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

$\backslash$

Step 1: Express $\backslash b_1 \backslash$ and $\backslash b_2 \backslash$

Here, $\backslash b_1 = [1, 0]^T \backslash$, $\backslash b_2 = [0, 1]^T \backslash$.

Step 2: Find coordinates of $\backslash b_1 \backslash$ in basis C

Solve:

$$\begin{cases} 1 = 2a_{11} + 1a_{12} \\ 0 = 1a_{11} + 3a_{12} \end{cases}$$

From the second equation:

$$0 = a_{11} + 3a_{12} \Rightarrow a_{11} = -3a_{12}$$

Substitute into the first:

$$1 = 2(-3a_{12}) + a_{12} = -6a_{12} + a_{12} = -5a_{12}$$

Thus:

$$a_{12} = -\frac{1}{5}$$

and

$$a_{11} = -3 \times \left(-\frac{1}{5}\right) = \frac{3}{5}$$

Step 3: Find coordinates of $\mathbf{b}_2$ in basis C

Solve:

$$\begin{cases} 0 = 2a_{21} + 1a_{22} \\ 1 = 1a_{21} + 3a_{22} \end{cases}$$

From the first:

$$0 = 2a_{21} + a_{22} \Rightarrow a_{22} = -2a_{21}$$

Substitute into the second:

$$1 = a_{21} + 3(-2a_{21}) = a_{21} - 6a_{21} = -5a_{21}$$

So:

$$a_{21} = -\frac{1}{5}$$

and

$$a_{22} = -2 \times \left(-\frac{1}{5}\right) = \frac{2}{5}$$

Step 4: Assemble the change-of-coordinates matrix

$$P_{C \rightarrow B} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

This matrix converts coordinates from basis B to basis C.

Applications of Change-of-coordinates Matrices

Understanding and computing change-of-coordinates matrices is crucial in various applications:

1. **Coordinate Transformation:** Switching between different coordinate systems simplifies computations in physics, engineering, and computer graphics.
2. **Diagonalization:** Changing bases to diagonalize matrices involves change-of-coordinates matrices.
3. **Linear Transformations:** Analyzing how linear transformations behave under different bases helps in understanding their structure.
4. **Eigenvalue Problems:** Changing bases can simplify the computation of eigenvalues and eigenvectors.

Summary and Key Takeaways

- To find the change-of-coordinates matrix from basis B to C in $\mathbb{R}^2$, express each vector of B in terms of C .
- The coefficients obtained from these expressions form the columns of the matrix $(P_{\{C \leftarrow B\}})$.
- This matrix allows seamless conversion of coordinate vectors between bases, facilitating various linear algebra operations.
- The process involves solving systems of linear equations, emphasizing the importance of understanding linear combinations and independence.
- Practical examples demonstrate the step-by-step approach, reinforcing theoretical understanding.

Final Thoughts

Mastering the process of finding change-of-coordinates matrices is essential for anyone studying linear algebra. It underpins many advanced topics, including matrix diagonalization, similarity transformations, and the analysis of linear systems. By understanding how to express basis vectors and solve the associated systems,

Frequently Asked Questions

Given bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for $\mathbb{R}^2$, how do you find the change-of-coordinates matrix from B to C ?

To find the change-of-coordinates matrix from B to C , express each basis vector of B in terms of the basis C , then arrange these coordinate vectors as columns of the matrix. The resulting matrix transforms coordinates from B to C .

What is the significance of the change-of-coordinates matrix from basis B to basis C in $\mathbb{R}^2$?

It allows you to convert coordinate representations of vectors from the B -basis to the C -basis, facilitating basis transformation and simplifying calculations in different coordinate systems.

If $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ are bases for $\mathbb{R}^2$, how do you compute the matrix from B to C explicitly?

First, find the coordinate vectors of each b_i expressed in basis C , then arrange these coordinate vectors as columns in a matrix. This matrix is the change-of-coordinates matrix from B to C .

What assumptions are made about bases B and C when calculating the change-of-coordinates matrix?

It is assumed that both B and C are bases for $\mathbb{R}^2$, meaning each set consists of two linearly independent vectors spanning $\mathbb{R}^2$, ensuring the matrices involved are invertible.

How does the change-of-coordinates matrix relate to the inverse matrix when converting from C back to B ?

The inverse of the change-of-coordinates matrix from B to C gives the change-of-coordinates matrix from C back to B , allowing conversion in both directions.

Can the change-of-coordinates matrix be used to transform any vector from B-coordinate form to C-coordinate form? How?

Yes. Multiply the vector's B-coordinate representation by the change-of-coordinates matrix from B to C to obtain the vector's C-coordinate form.

What steps are involved in finding the change-of-coordinates matrix if the bases are given explicitly?

1. Write each basis vector of B in terms of the standard basis.
2. Express these vectors in terms of C.
3. Form coordinate vectors relative to C.
4. Assemble these into a matrix; that matrix is from B to C.

Why is it important that both B and C are bases for $\mathbb{R}^2$ when computing the change-of-coordinates matrix?

Because bases are linearly independent sets that span $\mathbb{R}^2$, ensuring the matrices formed are invertible and the coordinate transformations are well-defined.

If the basis vectors are given numerically, how do you verify that the change-of-coordinates matrix is correct?

By multiplying the change-of-coordinates matrix by the coordinate vector of a basis vector in B, you should obtain the coordinate vector of that basis vector in C. Additionally, check the invertibility of the matrix.

Additional Resources

Understanding the Change-of-Coordinates Matrix Between Bases in $\mathbb{R}^2$

In the realm of linear algebra, the concept of changing coordinates between different bases is

fundamental to understanding the structure and transformations within vector spaces. When dealing with $\mathbb{R}^2$, the two-dimensional real vector space, the ability to transition smoothly from one basis to another is crucial for various applications—from solving systems of equations to geometric transformations. This article delves into the process of finding the change-of-coordinates matrix from a basis $B = \{b_1, b_2\}$ to another basis $C = \{c_1, c_2\}$, exploring the theoretical foundation, computational steps, and interpretative insights involved in this fundamental operation.

Foundations: Bases and Coordinates in $\mathbb{R}^2$

What Are Bases in $\mathbb{R}^2$?

A basis in $\mathbb{R}^2$ is a set of two linearly independent vectors that span the entire space. Specifically, for $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$, each set must satisfy two conditions:

1. Linear Independence: Neither vector is a scalar multiple of the other.
2. Spanning $\mathbb{R}^2$: Any vector in $\mathbb{R}^2$ can be expressed as a linear combination of the basis vectors.

The significance of choosing different bases lies in their ability to simplify problems, adapt to geometric or algebraic contexts, or facilitate computations.

Coordinates of a Vector Relative to a Basis

Given a basis B , any vector $v \in \mathbb{R}^2$ can be uniquely written as:

$$v = \alpha_1 b_1 + \alpha_2 b_2$$

where $(\alpha_1, \alpha_2 \in \mathbb{R})$. The ordered pair (α_1, α_2) is called the coordinate vector of v relative to B , denoted as $[v]_B$.

Similarly, relative to C , the same vector v has coordinate vector $[v]_C$. Transitioning between these coordinate representations involves a linear transformation characterized by the change-of-coordinates matrix.

Change-of-Coordinates Matrix: Conceptual Overview

Definition and Purpose

The change-of-coordinates matrix from basis B to basis C , often denoted as $(P_{C \leftarrow B})$, is a matrix that transforms the coordinate vector of any vector v expressed in basis B into its coordinate vector expressed in basis C :

$$[v]_C = P_{C \leftarrow B} [v]_B$$

This matrix encapsulates the relationship between the two bases and provides a computational tool for converting coordinates efficiently.

Why Is This Matrix Important?

- **Coordinate Transformation:** It simplifies the process of switching between different coordinate systems.
- **Linear Transformations:** It aids in understanding how linear transformations behave under different

bases.

- Applications: Used extensively in computer graphics, engineering, physics, and more, where changing perspectives or coordinate systems is routine.

Computational Approach: Deriving the Change-of-Coordinates Matrix

To compute $(P_{C \leftarrow B})$, we need to understand how the basis vectors of B are expressed relative to C .

Step 1: Express Basis Vectors of B in Terms of C

For each basis vector (b_i) , find its coordinate vector relative to C :

$$[b_i]_C = \text{coordinates of } b_i \text{ in basis } C$$

This involves solving the linear equations:

$$b_i = \lambda_1 c_1 + \lambda_2 c_2$$

for (λ_1, λ_2) .

Step 2: Assemble the Change-of-Coordinates Matrix

Once the coordinate vectors $[b_1]_C$ and $[b_2]_C$ are found, they form the columns of the matrix

$\mathcal{P}_{C \leftarrow B}$:

$$\mathcal{P}_{C \leftarrow B} = \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix}$$

This matrix, when multiplied by the coordinate vector of a vector v in basis B , yields the coordinate vector in basis C .

Step 3: Practical Calculation

Suppose the vectors are explicitly given:

$$- \mathbf{b}_1 = (b_{11}, b_{12})$$

$$- \mathbf{b}_2 = (b_{21}, b_{22})$$

and

$$- \mathbf{c}_1 = (c_{11}, c_{12})$$

$$- \mathbf{c}_2 = (c_{21}, c_{22})$$

Express $\mathbf{b}_1$ as:

$$\mathbf{b}_1 = x_{11} \mathbf{c}_1 + y_{11} \mathbf{c}_2$$

Similarly, express $\mathbf{b}_2$:

$$\mathbf{b}_2 = x_{21} \mathbf{c}_1 + y_{21} \mathbf{c}_2$$

The coefficients form the columns of the change-of-coordinates matrix:

$$\mathcal{P}_{C \leftarrow B} = \begin{bmatrix} x_{11} & x_{21} \\ y_{11} & y_{21} \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{21} \\ y_{11} & y_{21} \end{bmatrix}$$

Example: Explicit Calculation with Given Bases

Suppose the bases are:

$$B = \{b_1 = (1, 0), b_2 = (0, 1)\}$$

$$C = \{c_1 = (2, 1), c_2 = (1, 3)\}$$

Step 1: Express each B basis vector in terms of C.

- For $b_1 = (1, 0)$:

Find (α, β) such that:

$$(1, 0) = \alpha (2, 1) + \beta (1, 3)$$

which leads to the system:

$$\begin{cases} 2\alpha + \beta = 1 \\ \alpha + 3\beta = 0 \end{cases}$$

Solve:

From the second equation:

$$\alpha = -3\beta$$

Plug into the first:

$$2(-3\beta) + \beta = 1 \Rightarrow -6\beta + \beta = 1 \Rightarrow -5\beta = 1 \Rightarrow \beta = -\frac{1}{5}$$

Then,

$$\alpha = -3 \times -\frac{1}{5} = \frac{3}{5}$$

Thus,

$$[b_1]_C = \left(\frac{3}{5}, -\frac{1}{5} \right)$$

- For $(b_2 = (0, 1))$:

Find (γ, δ) such that:

$$(0, 1) = \gamma(2, 1) + \delta(1, 3)$$

which leads to:

$$\begin{cases} 2\gamma + \delta = 0 \\ \gamma + 3\delta = 1 \end{cases}$$

$\end{cases}$

$\]$

Solve:

From the first:

$$\delta = -2\gamma$$

Substitute into the second:

$$\begin{aligned} \gamma + 3(-2\gamma) &= 1 \Rightarrow \gamma - 6\gamma = 1 \Rightarrow -5\gamma = 1 \\ \Rightarrow \gamma &= -\frac{1}{5} \end{aligned}$$

Then,

$$\delta = -2 \times -\frac{1}{5} = \frac{2}{5}$$

So,

$$[b_2]_C = \left(-\frac{1}{5}, \frac{2}{5} \right)$$

Step 2: Form the matrix:

$$[P_C \leftarrow B] = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

This matrix transforms B-coordinates to C-coordinates.

Interpreting the Change-of-Coordinates Matrix

Properties and Significance

- Invertibility: Since both B and C are bases, $(P_{C \leftarrow B})$ is invertible. Its inverse transforms C -coordinates back to B -coordinates.
- Determinant: The determinant of $(P_{C \leftarrow B})$ indicates the area scaling factor between bases. A determinant of 1 means no area distortion; less than 1 indicates compression, greater than 1 indicates expansion.
- Geometric Meaning: The matrix encapsulates how basis vectors of B are “reshaped” or “reoriented” when expressed relative to C .

Applications in Practice

- Coordinate Conversion: Quickly switch between coordinate systems in computational applications.
- Linear Transformations: Understand how a change of basis affects linear operators, such as rotations or reflections.
- Geometric Analysis: Compute areas, angles, and other geometric metrics with respect to different bases.

Conclusion: The Broader Context of Basis Transformation

Understanding how to compute and interpret the change-of-coordinates matrix from one basis to another is a cornerstone of linear algebra with extensive applications. Whether in pure mathematics, physics, computer

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