Let {bn} Be A Sequence Such That Bn =n1/n. Show That Bn Is Decreasing By Provingthat Following:Prove

Let {bn} Be A Sequence Such That Bn = n^{1/n}. Show That Bn Is Decreasing By Proving That Following: Prove

Introduction

Understanding the behavior of sequences is fundamental in mathematical analysis, especially in the study of limits, convergence, and monotonic properties. One interesting sequence is defined as $b_n = n^{1/n}$, which involves the n-th root of n. This sequence appears in various contexts, such as limits involving roots and exponential functions.

In this article, we will explore the properties of the sequence $b_n = n^{1/n}$, specifically focusing on demonstrating that this sequence is decreasing for sufficiently large n. To do this, we will establish the necessary mathematical framework, employ derivative tests, and provide detailed proofs to confirm the decreasing nature of b_n.

Understanding the Sequence $b_n = n^{1/n}$

Before diving into the proof, it is important to understand the structure and behavior of the sequence $b_n = n^{1/n}$. Let's analyze its general characteristics:

- Definition: For each positive integer n,
$$
b_n = n^{1/n}
$$

- Interpretation: This can be viewed as the n-th root of n. As n increases, the root tends to "dilute" the growth of n, leading to interesting asymptotic behaviors.

- Initial Values:
- $b_1 = 1^{1/1} = 1$
- $b_2 = 2^{1/2} \approx 1.414$
- $b_3 = 3^{1/3} \approx 1.442$
- $b_4 = 4^{1/4} \approx 1.414$
- $b_5 = 5^{1/5} \approx 1.379$

Observing these values, we notice that b_n increases from $n=1$ to some maximum point, then starts decreasing. Our goal is to rigorously prove that beyond a certain point, the sequence is decreasing.

Reformulating the Sequence Using Logarithms

To analyze the monotonicity of b_n, it is often easier to consider its logarithm. Define:

$$
a_n = \ln b_n = \ln (n^{1/n}) = \frac{1}{n} \ln n
$$

The sequence b_n is decreasing if and only if $a_n = \frac{\ln n}{n}$ is decreasing, because the exponential function e^{a_n} is strictly increasing.

Thus, our task reduces to examining the monotonicity of $a_n = \frac{\ln n}{n}$ for $n \geq 1$.

Analyzing the Sequence $a_n = \frac{\ln n}{n}$

The function a_n is defined for $n \geq 1$, with the domain being the natural numbers. To analyze its monotonicity, we consider the corresponding continuous function:

$$
f(x) = \frac{\ln x}{x}, \quad x > 0
$$

Understanding the behavior of $f(x)$ allows us to infer the behavior of the sequence a_n, especially for large n.

Derivative of $f(x)$

Calculating the derivative:

$$
f'(x) = \frac{(1/x) \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}
$$

- For $x > 0$, $x^2 > 0$, so the sign of $f'(x)$ depends on $1 - \ln x$.

- Critical point occurs when $f'(x) = 0$:

$$
1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e

\]

Thus, $f(x)$ has a maximum at $x = e$.

Monotonic Behavior of $f(x)$

- For $x < e$:
$$
\ln x < 1 \Rightarrow 1 - \ln x > 0 \Rightarrow f'(x) > 0
$$
So, $f(x)$ is increasing on $(0, e)$.

- For $x > e$:
$$
\ln x > 1 \Rightarrow 1 - \ln x < 0 \Rightarrow f'(x) < 0
$$
So, $f(x)$ is decreasing on (e, ∞).

Since $a_n = \frac{\ln n}{n}$ approximates $f(x)$, the sequence a_n increases for $n < e$ and decreases for $n > e$.

Given that $e \approx 2.718$, for natural numbers:
- $a_1 = \ln 1 / 1 = 0$
- $a_2 = \ln 2 / 2 \approx 0.346$
- $a_3 = \ln 3 / 3 \approx 0.366$
- $a_4 = \ln 4 / 4 \approx 0.346$
- $a_5 = \ln 5 / 5 \approx 0.322$

We observe that a_n increases from $n=1$ to $n=3$, then begins decreasing from $n=4$ onwards.

Summary:
- $a_1 = 0$
- $a_2 \approx 0.346$
- $a_3 \approx 0.366$ (maximum at $n=3$)
- $a_4 \approx 0.346$
- $a_5 \approx 0.322$

Therefore, for $n \geq 3$, a_n is decreasing.

Proving that $b_n = n^{1/n}$ Is Decreasing for $n \geq 3$

Since $b_n = e^{a_n}$, and the exponential function is strictly increasing, the decreasing nature of a_n for $n \geq 3$ implies b_n is decreasing for $n \geq 3$.

Formal proof:

1. Identify the domain where a_n decreases:

From the derivative analysis, a_n decreases for $n > e$, i.e., for all $n \geq 3$.

2. Show that $a_{n+1} < a_n$ for $n \geq 3$:

Since $a_n = \frac{\ln n}{n}$, consider the difference:

$$
a_{n+1} - a_n = \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n}
$$

3. Prove $a_{n+1} < a_n$ for $n \geq 3$:

Cross-multiplied:

$$
(\ln(n+1)) \cdot n - (\ln n) (n+1) < 0
$$

Simplify:

$$
n \ln(n+1) - (n+1) \ln n < 0
$$

Rearranged:

$$
n \ln(n+1) < (n+1) \ln n
$$

4. Numerical verification for $n \geq 3$:

- For $n=3$:

Left side:

$$
3 \ln 4 \approx 3 \times 1.386 = 4.158
$$

Right side:

$$
4 \ln 3 \approx 4 \times 1.0986 = 4.394
$$

\]

Since $4.158 < 4.394$, the inequality holds.

- For larger n, similar calculations show the inequality continues to hold, confirming a_n decreases beyond $n=3$.

5. Conclusion:

Because a_n decreases for $n \geq 3$ and $b_n = e^{a_n}$, the sequence b_n is decreasing for all $n \geq 3$.

Behavior of the Sequence for Small Values of n

It is noteworthy that:

- $b_1 = 1$
- $b_2 \approx 1.414$
- $b_3 \approx 1.442$
- $b_4 \approx 1.414$

The sequence increases from $n=1$ to $n=3$, reaching a maximum at $n=3$, then decreases for $n \geq 4$. This aligns with

Frequently Asked Questions

What is the sequence {bn} defined as in the problem?

The sequence {bn} is defined by bn = $n^{1/n}$, meaning each term is the n-th root of n.

What does it mean for the sequence {bn} to be decreasing?

A sequence {bn} is decreasing if for all n, bn+1 ≤ bn, meaning each subsequent term is less than or equal to the previous one.

How can we prove that the sequence {bn} = $n^{1/n}$ is decreasing?

We can prove it by showing that the function f(n) = $n^{1/n}$ is decreasing for n ≥ 1, typically by analyzing its derivative or comparing successive terms.

What is the common approach to prove that a sequence is decreasing?

A common approach is to compare consecutive terms, i.e., show that $b_{n+1} \leq b_n$ for all n, or to analyze the function's derivative to determine its monotonicity.

How can we use logarithms to analyze the sequence {bn}?

By taking the natural logarithm, $\ln(b_n) = (1/n) \ln(n)$. Analyzing the behavior of $\ln(b_n)$ can help determine whether $\{b_n\}$ is decreasing.

What is the derivative of the function f(n) = n^{1/n}?

The derivative of $f(n) = n^{1/n}$ can be found by first taking $\ln(f(n)) = (1/n) \ln(n)$, then differentiating and analyzing the sign of the derivative to determine decreasing behavior.

What is the significance of showing that the derivative of ln(f(n)) is negative?

If the derivative of $\ln(f(n))$ is negative, then $\ln(f(n))$ is decreasing, which implies that $f(n) = n^{1/n}$ is decreasing for the corresponding domain.

Can you outline the steps to prove that bn = n^{1/n} is decreasing?

Yes. First, take the natural logarithm: $\ln(b_n) = (1/n) \ln(n)$. Next, differentiate with respect to n to find the sign of the derivative. If the derivative is negative for $n \geq 1$, then b_n is decreasing. Alternatively, compare successive terms b_n and b_{n+1} to show $b_{n+1} \leq b_n$.

What is the value of bn at n=1, and how does it help in the proof?

At $n=1$, $b_n = 1^{1/1} = 1$. Showing that the sequence decreases from this point onward helps establish that $\{b_n\}$ is decreasing for all $n \geq 1$.

What conclusion can be drawn once it is proven that {bn} = n^{1/n} is decreasing?

Once proven, we conclude that the sequence $\{b_n\}$ is decreasing for all $n \geq 1$, meaning each subsequent term is less than or equal to the previous one, confirming the behavior of the sequence as n increases.

Additional Resources

Sequence analysis is a fundamental aspect of mathematical exploration, providing insights into the behavior of mathematical functions and their applications across various disciplines. Among the myriad of sequences studied, the sequence $\{b_n\}$ defined by $b_n = n^{1/n}$ offers intriguing properties worth examining, particularly its monotonic behavior. This article delves into the sequence $\{b_n\}$, demonstrating that it is decreasing for sufficiently large n by rigorously proving the related inequalities and properties that govern its trend.

Introduction to the Sequence $\{b_n\}$

Definition and Basic Understanding

Let us consider the sequence $\{b_n\}$, where

$$
b_n = n^{1/n}
$$

for each positive integer n. At first glance, this sequence appears to involve the n-th root of n, a function that combines exponential growth with root extraction. The sequence is well-defined for all $n \in \mathbb{N}$, and examining its behavior involves understanding how the expression $n^{1/n}$ varies as n increases.

Intuitive Behavior:

- For small values of n, the sequence behaves differently. For example:
- $b_1 = 1^{1/1} = 1$
- $b_2 = 2^{1/2} \approx 1.414$
- $b_3 = 3^{1/3} \approx 1.442$
- $b_4 = 4^{1/4} = \sqrt{2} \approx 1.414$

- As n increases further, the sequence initially increases, reaching a maximum, before eventually decreasing.

Understanding whether $\{b_n\}$ is increasing or decreasing, especially for large n, is crucial for analyzing its limit and other properties.

Exploring the Monotonicity of $\{b_n\}$

Why Monotonicity Matters

Determining whether the sequence $\{b_n\}$ is decreasing or increasing helps in understanding its limit behavior and convergence properties. For sequences involving roots or exponential functions, the monotonicity often hinges on the interplay between exponential growth and root extraction. Specifically, knowing that $\{b_n\}$ is decreasing after a certain point allows mathematicians to conclude that the sequence converges, and to identify its limit.

Key Objective: Proving $\{b_n\}$ Is Decreasing for Large n

The main goal of this discussion is to demonstrate that $\{b_n\}$ is decreasing beyond some threshold value of n. Formally, we seek to show that:

$$
b_{n+1} < b_n \quad \text{for sufficiently large } n
$$

which equivalently means:

$$
(n+1)^{1/(n+1)} < n^{1/n}
$$

or, expressed differently, the sequence's terms get smaller as n increases past a certain point.

Analytical Approach to Monotonicity

Transforming the Sequence for Easier Analysis

A common strategy in analyzing sequences involving exponents and roots is to take the natural logarithm, simplifying the comparison:

$$
b_n = n^{1/n}
$$

$$
\Rightarrow \ln b_n = \frac{1}{n} \ln n
$$

This logarithmic transformation turns the problem into analyzing the sequence:

\[
a_n = \frac{\ln n}{n}
\]

The monotonic behavior of \(\{b_n\} \) corresponds to that of \(\{a_n\} \), because the exponential function is strictly increasing. Specifically, \(\{b_n\} \) is decreasing if and only if \(\{a_n\} \) is decreasing, i.e.,

\[
a_{n+1} < a_n
\]

which amounts to analyzing:

\[
\frac{\ln(n+1)}{n+1} < \frac{\ln n}{n}
\]

for sufficiently large \(n \).

Mathematical Analysis of \(a_n = \frac{\ln n}{n} \)

To establish whether \(a_n \) decreases as \(n \) increases, we examine the difference:

\[
a_{n+1} - a_n = \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n}
\]

Calculating this difference helps determine the monotonicity:

\[
a_{n+1} - a_n = \frac{\ln(n+1)}{n+1} - \frac{\ln n}{n}
\]

Expressed over a common denominator:

\[
a_{n+1} - a_n = \frac{n \ln(n+1) - (n+1) \ln n}{n(n+1)}
\]

The sign of the numerator determines whether the sequence \(a_n \) is increasing or decreasing:

\[
N_n = n \ln(n+1) - (n+1) \ln n

$$

Objective: Show $N_n < 0$ for sufficiently large n, implying $a_{n+1} < a_n$, and consequently, $b_{n+1} < b_n$.

Analyzing N_n: The Key Inequality

Let’s analyze N_n:

$$
N_n = n \ln(n+1) - (n+1) \ln n
$$

Rewrite as:

$$
N_n = n \ln(n+1) - n \ln n - \ln n
$$
$$
= n (\ln(n+1) - \ln n) - \ln n
$$

Recall that:

$$
\ln(n+1) - \ln n = \ln \left(\frac{n+1}{n} \right) = \ln \left(1 + \frac{1}{n} \right)
$$

Thus:

$$
N_n = n \ln \left(1 + \frac{1}{n} \right) - \ln n
$$

Using the Taylor expansion of the natural logarithm for small x:

$$
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots
$$

for $x = 1/n$:

$$
\ln \left(1 + \frac{1}{n} \right) \approx \frac{1}{n} - \frac{1}{2 n^2}
$$

Multiplying through by n:

$$
n \ln \left(1 + \frac{1}{n} \right) \approx 1 - \frac{1}{2 n}
$$

So,

$$
N_n \approx 1 - \frac{1}{2 n} - \ln n
$$

For large n, $\ln n$ dominates, and since $\ln n \to \infty$, it follows that:

$$
N_n \to -\infty
$$

meaning, for sufficiently large n:

$$
N_n < 0
$$

which implies:

$$
a_{n+1} < a_n
$$

and consequently,

$$
\ln b_{n+1} < \ln b_n
$$
$$
\Rightarrow b_{n+1} < b_n
$$

for sufficiently large n. This confirms that the sequence $\{b_n\}$ is decreasing beyond a certain point.

Formal Proof and Limitations

Establishing the Threshold n

Our approximate analysis indicates that for large n, a_n decreases, and thus b_n decreases. To be rigorous, we can analyze the inequality:

\[
a_{n+1} < a_n
\]
\[
\Rightarrow N_n < 0
\]
\[
\Rightarrow n \ln \left(1 + \frac{1}{n} \right) < \ln n
\]

Using the fact that:

\[
\ln \left(1 + \frac{1}{n} \right) < \frac{1}{n}
\]

for all \(n \geq 1 \) (since \(\ln(1+x) < x \) for \(x > 0 \)), we get:

\[
n \cdot \frac{1}{n} = 1 < \ln n
\]

which holds when:

\[
\ln n > 1 \quad \Rightarrow \quad n > e
\]

Thus, for all \(n > e \approx 2.718 \), the inequality:

\[
a_{n+1} < a_n
\]

holds, confirming that \(\{a_n\} \), and consequently \(\{b_n\} \), becomes decreasing for all \(n \geq 3 \

Let Bn Be A Sequence Such That Bn N1 N Show That Bn Is Decreasing By Provingthat Following Prove

Let Bn Be A Sequence Such That Bn N1 N Show That Bn Is Decreasing By Provingthat Following Prove

Related Articles

- In The Image, Point A Marks The Center Of The Circle. Which Two Lengths Must Form A Ratio Of 1:2? A.
- Melanie Cooper Was Hired To Serve As Chief Executive Officer Of Lexington Home For The Aged, Inc., A
- Imagine A Therapist Is Interested In Measuring Her Client's Propensity For Taking Risks. She Decides