

Let $D = \{d_1, d_2, \dots, d_s\}$ And $F = \{f_1, \dots, f_s\}$ Be Bases For A Vector Space V . And Suppose $F = (a)$ Find The

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Introduction

Understanding the structure of vector spaces is fundamental in linear algebra. When working with different bases, it becomes crucial to understand how vectors transform from one basis to another and how to express vectors in terms of these bases. In this article, we will explore the problem of finding the coordinates of a vector relative to different bases, given a specific change of basis matrix, denoted as $(F|I = (a))$. We will walk through the theoretical background, practical methods, and example computations to clarify this concept.

Foundations of Vector Spaces and Bases

What is a Vector Space?

A vector space (V) over a field (F) (such as the real numbers $(\mathbb{R})$ or complex numbers $(\mathbb{C})$) is a collection of objects called vectors, equipped with operations of vector addition and scalar multiplication that satisfy specific axioms.

Bases of a Vector Space

A basis of (V) is a set of vectors $(\{d_1, d_2, \dots, d_s\})$ such that:

- The vectors are linearly independent.
- The span of these vectors equals (V) .

Similarly, another basis $(\{f_1, f_2, \dots, f_s\})$ also spans (V) , but the vectors may be different.

Coordinate Representation

Any vector $(v \in V)$ can be uniquely expressed as a linear combination of the basis vectors:

$$v = c_1 d_1 + c_2 d_2 + \dots + c_s d_s$$

or

(v)

$$v = c'_1 f_1 + c'_2 f_2 + \cdots + c'_s f_s$$

where $\{c_i\}$ and $\{c'_i\}$ are coordinate vectors relative to the bases $\{D\}$ and $\{F\}$, respectively.

Change of Basis and Transformation Matrices

The Concept of Change of Basis

When switching from one basis to another, the coordinates of a vector change accordingly. The key tool for this process is the change of basis matrix.

Defining the Change of Basis Matrix F_{-I}

Suppose $F_{-I} = (a)$ is the matrix that relates the bases $\{D\}$ and $\{F\}$. Typically, this matrix is constructed such that:

$$\text{For a vector } v, \quad [v]_F = F_{-I} [v]_D$$

where:

- $[v]_D$ is the coordinate vector of v relative to basis $\{D\}$,
- $[v]_F$ is the coordinate vector of v relative to basis $\{F\}$,
- F_{-I} is the matrix of transition from $\{D\}$ to $\{F\}$.

Properties of F_{-I}

- F_{-I} is invertible because both $\{D\}$ and $\{F\}$ are bases.
- The inverse matrix F_{-I}^{-1} transforms coordinates from basis $\{F\}$ back to $\{D\}$.

The Problem: Finding Coordinates with Given Data

Given the bases $D = \{d_1, d_2, \dots, d_s\}$, $F = \{f_1, f_2, \dots, f_s\}$, and the transformation matrix $F_{-I} = (a)$, the problem is:

"Suppose $F_{-I} = (a)$. Find the coordinates of a vector v in basis $\{D\}$ or $\{F\}$."

In many cases, the problem might specify a particular vector v or ask to compute the image of a vector under the change of basis. Let's explore the typical steps involved.

Step-by-Step Solution Approach

1. Understanding the Transformation Matrix (F_I)

- Recognize that (F_I) encodes how vectors in basis (D) are represented in basis (F) .
- If $(F_I = (a))$, this indicates the matrix's entries, which will be used to convert coordinates.

2. Expressing a Vector in One Basis

Suppose you have a vector (v) with known coordinates:

- $([v]_D = \mathbf{c}_D)$
- $([v]_F = \mathbf{c}_F)$

Given these, the relationship is:

$$[\mathbf{c}_F = F_I \mathbf{c}_D]$$

or equivalently,

$$[\mathbf{c}_D = F_I^{-1} \mathbf{c}_F]$$

3. Computing the Coordinates

- To find the coordinate vector of (v) in basis (D) , if you know its coordinates in basis (F) , multiply by (F_I^{-1}) .
- To find the coordinate vector in basis (F) , multiply its (D) -basis coordinates by (F_I) .

4. Practical Example

Suppose:

- $(F_I = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{bmatrix})$
- The vector (v) has known coordinates in basis (D) : $(\mathbf{c}_D)$.

Then:

$$[\boxed{\mathbf{c}_F = F_I \mathbf{c}_D}]$$

and vice versa.

Applications and Examples

Example 1: Basic Computation

Suppose:

$$F_I = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

and the coordinate vector in (D) basis:

$$\mathbf{c}_D = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Find the coordinate vector in the (F) basis:

$$\begin{aligned} \mathbf{c}_F = F_I^{-1} \mathbf{c}_D &= \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 4 + 0 \times 5 \\ 1 \times 4 + 3 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 4 + 15 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \end{bmatrix} \end{aligned}$$

Thus, the coordinates of (v) in basis (F) are $(\{8, 19\})$.

Example 2: Inverse Transformation

Suppose you know a vector's coordinates in (F) basis and want to find its coordinates in (D) . Then:

$$\mathbf{c}_D = F_I^{-1} \mathbf{c}_F$$

Compute the inverse of (F_I) :

$$F_I^{-1} = \frac{1}{\det(F_I)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

If the determinant is non-zero, this inverse exists, and you can perform the calculation accordingly.

Significance in Linear Algebra and Applications

Understanding how to convert vector coordinates between different bases has numerous applications:

- Coordinate transformations: Essential in computer graphics, robotics, and physics.
- Diagonalization and eigenvalues: Changing bases to simplify matrix representations.
- Data analysis: Transforming data into different coordinate systems for analysis.
- Solving linear systems: Utilizing basis changes to simplify computations.

Summary

This article has provided a comprehensive overview of the process of finding vector coordinates relative to different bases in a vector space, given a change of basis matrix (F_{-1}) . The key points include:

- Recognizing the roles of bases (D) and (F) .
- Understanding the significance of the transformation matrix (F_{-1}) .
- Applying matrix multiplication and inversion to convert coordinates.
- Practical computation methods with illustrative examples.

Mastering these concepts is fundamental for advanced studies in linear algebra and its applications across various scientific and engineering fields.

Final Tips for Practitioners

- Always verify that your matrix (F_{-1}) is invertible before attempting to find $(F_{-1})^{-1}$.
- Keep track of which basis your coordinates are expressed in to avoid confusion.
- Practice with concrete examples to build intuition on basis transformations.
- Use computational tools like MATLAB, NumPy, or graphing calculators for complex calculations.

By mastering basis transformations and change of basis matrices, you will enhance your ability to analyze and manipulate vector spaces efficiently in both theoretical and applied contexts.

Frequently Asked Questions

Given bases $D = \{d_1, d_2, d_s\}$ and $F = \{f_1, f_2, f_3\}$ for a vector space V , how do you find the transition matrix from D to F ?

To find the transition matrix from D to F , express each vector in D as a linear combination of vectors in F . The coefficients form the columns of the transition matrix. Specifically, write d_i as a combination of f_i 's, and place these coefficients as columns in the matrix.

If D and F are bases for V and you have a vector v in V , how can you find its coordinate vector with respect to both bases?

First, express v as a linear combination of the basis D to get its coordinate vector relative to D . Then, convert that coordinate vector into the F basis by multiplying with the inverse of the transition matrix from D to F , obtaining v 's coordinates in F .

What is the significance of the transition matrix between two

bases in a vector space?

The transition matrix allows us to convert coordinate representations of vectors from one basis to another, facilitating changes of basis and understanding how vectors transform under basis changes.

How do you verify if a set of vectors forms a basis for a vector space V ?

Check if the vectors are linearly independent and span V . This can be confirmed by ensuring the vectors are linearly independent or by their matrix having full rank equal to the dimension of V .

Given bases D and F , how can you find the coordinates of a vector v in basis F if you know its coordinates in basis D ?

Multiply the coordinate vector of v in D by the transition matrix from D to F to obtain the coordinates of v in basis F .

What is the process to find the basis vectors of a subspace given two bases D and F ?

Identify the vectors common to both bases or express one basis in terms of the other. For subspaces, find the vectors that satisfy the subspace criteria and are linear combinations of the original bases.

How is the concept of bases related to the dimension of a vector space?

The number of vectors in any basis of a vector space V defines its dimension. All bases of V have the same number of elements, which equals the dimension.

What are the conditions for two sets of vectors to be bases of the same vector space?

Both sets must be linearly independent and span the entire vector space V .

How do you interpret the transition matrix when moving from basis D to basis F ?

The transition matrix describes how to convert coordinate vectors from basis D to basis F , effectively changing the perspective from one basis to another within the same vector space.

In the context of bases D and F for V , what does it mean if the transition matrix from D to F is the identity matrix?

It means that bases D and F are the same; the vectors are identical in both bases, and the coordinate representations are directly comparable without transformation.

Additional Resources

Understanding Vector Spaces, Bases, and Coordinate Systems: An Expert Breakdown

When delving into the fascinating world of linear algebra, the concepts of vector spaces, bases, and coordinate representations form the backbone of much advanced mathematical theory and practical application. Whether you're a student, a researcher, or a professional in engineering or data science, grasping these core ideas is essential. This article offers a comprehensive exploration of bases in vector spaces, highlighting the significance of basis sets, their properties, and the process of coordinate transformation, all through an expert lens.

Introduction to Vector Spaces and Bases

Before diving into the specifics, it's crucial to understand what a vector space is and why bases matter.

What is a Vector Space?

A vector space over a field $\mathbb{F}$ (like the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$) is a collection of objects called vectors, equipped with two operations:

- Vector addition: Combining two vectors to produce a third.
- Scalar multiplication: Multiplying a vector by a scalar from $\mathbb{F}$.

These operations adhere to axioms such as commutativity, associativity, distributivity, and the existence of additive identities and inverses.

The Importance of Bases

A basis of a vector space V is a minimal set of vectors that:

1. Span V : Any vector in V can be expressed as a linear combination of the basis vectors.
2. Are linearly independent: No basis vector can be written as a linear combination of the others.

The choice of basis is fundamental because it provides a coordinate system for V , enabling efficient calculations, transformations, and understanding of the structure of the space.

Defining the Sets D and F as Bases

In this discussion, we are given:

- Set D: $D = \{d_1, d_2, \dots, d_s\}$

- Set $F: \{f_1, f_2, \dots, f_t\}$

and both are bases for the vector space V . This implies:

- Both D and F are linearly independent.
- Each set spans V .

Significance of Having Multiple Bases

Having multiple bases is standard in linear algebra because it offers flexibility. Different bases can simplify particular problems, reveal different properties, or facilitate coordinate transformations.

- Change of basis: Moving from one basis to another involves a transition matrix, which encodes how vectors are represented differently relative to each basis.
- Coordinate representation: Any vector $v \in V$ can be expressed uniquely in terms of either basis.

Expressing Vectors in Different Bases: The Core Concept

Suppose you are given a vector $v \in V$. Its representations relative to bases D and F are:

- Coordinates in D : $[v]_D = (a_1, a_2, \dots, a_s)$, such that

$$v = a_1 d_1 + a_2 d_2 + \dots + a_s d_s$$

- Coordinates in F : $[v]_F = (b_1, b_2, \dots, b_t)$, such that

$$v = b_1 f_1 + b_2 f_2 + \dots + b_t f_t$$

The transformation from one coordinate system to another is a fundamental process in linear algebra, essential for applications like change of basis, diagonalization, and principal component analysis.

Finding the Transition Matrix and Coordinate Relations

The Transition Matrix

The key to moving between the coordinate systems of D and F is the transition matrix $T_{F \leftarrow D}$. This matrix converts the coordinate vector of v relative to D into the coordinate vector relative to F .

- Constructing $T_{F \leftarrow D}$:

1. Express each basis vector (d_i) in terms of (F) :

$$d_i = \sum_{j=1}^t c_{ji} f_j$$

2. Collect the coefficient vectors $(c_{1i}, c_{2i}, \dots, c_{ti})$ as columns to form the matrix:

$$T_{F \rightarrow D} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1s} \\ c_{21} & c_{22} & \dots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t1} & c_{t2} & \dots & c_{ts} \end{bmatrix}$$

3. This matrix maps (v_D) to (v_F) :

$$v_F = T_{F \rightarrow D} \times v_D$$

The Inverse Transition

Similarly, the inverse matrix $(T_{D \rightarrow F} = (T_{F \rightarrow D})^{-1})$ allows conversion from (F) -coordinates back to (D) -coordinates.

Adopting the Process: From Theory to Practice

Suppose you are tasked with finding the coordinate vector of a new vector (v) relative to basis (F) , given its coordinates in (D) . The step-by-step procedure involves:

1. Identify (v_D) : The coordinate vector of (v) in basis (D) .
2. Compute (v_F) : Multiply (v_D) by the transition matrix $(T_{F \rightarrow D})$.
3. Interpretation: The resulting vector (v_F) provides the coordinates of (v) in the basis (F) .

This process is vital when working with different coordinate systems, simplifying complex calculations, or transforming data representations in applied contexts.

Practical Application: The Significance of Basis Choice

The choice of basis can dramatically influence computational efficiency and clarity:

- Simplifying calculations: Choosing a basis aligned with the problem can make the representation of vectors and linear transformations much simpler.
- Diagonalization: For linear operators, selecting an eigenbasis (a special basis where the operator

acts as scalar multiplication) simplifies analysis.

- Data Science: Principal component analysis (PCA) involves transforming data into a basis where variance is maximized along axes, effectively changing the basis for better interpretability.

Summary of Key Concepts

- Vector space (V) : A collection of vectors satisfying axioms of addition and scalar multiplication.
- Bases (D) and (F) : Sets of vectors spanning (V) , each linearly independent.
- Coordinate vectors: Unique representations of any $(v \in V)$ relative to a basis.
- Transition matrix: The core tool to convert coordinates from one basis to another.
- Change of basis: Facilitated by the transition matrix and its inverse.

Conclusion: Mastering Bases for Advanced Linear Algebra

Understanding the relationship between different bases, the process of expressing vectors in various coordinate systems, and the role of transition matrices forms a cornerstone of linear algebra. Whether you're working on theoretical problems or practical applications like computer graphics, data analysis, or engineering systems, mastery over these concepts enables precise, efficient, and insightful manipulation of vector spaces.

By carefully selecting bases and understanding how to transition between them, you unlock a powerful toolkit that simplifies complex transformations, reveals underlying structure, and enhances computational efficiency. The seemingly abstract concepts of bases and coordinate transformations are, in fact, fundamental to many modern technological and scientific advances.

In essence, mastering the use of different bases and their transition matrices is like having a universal translation guide in the language of vectors—allowing you to seamlessly navigate complex spaces and unlock their full potential.

Let $D = \{d_1, d_2, \dots, d_s\}$ and $F = \{f_1, f_2, \dots, f_s\}$ be bases for a vector space V and suppose T is a linear transformation from V to V . Find the matrix $[T]_{F \leftarrow D}$.

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