Let R Be A Notherian Ring And Let F Be A Non-zero Element In R. (a) Show That Dim R Max{dim R/P:P Is

Understanding Noetherian Rings and Their Dimensions

In algebraic geometry and commutative algebra, the concept of Noetherian rings plays a pivotal role. They serve as foundational building blocks for many structures and theories, especially when analyzing algebraic varieties, schemes, and modules. An essential aspect of studying Noetherian rings is understanding their Krull dimension, which measures the "height" or "size" of their prime spectrum. This article explores the properties of Noetherian rings, the behavior of their prime ideals, and how the Krull dimension interacts with elements within the ring, focusing on a specific problem involving the dimension of the ring and the prime ideals containing a non-zero element \(F\).

Defining Noetherian Rings and Krull Dimension

What Is a Noetherian Ring?

A ring \(R\) is called Noetherian if it satisfies the ascending chain condition (ACC) on ideals. This means:

- Every increasing sequence of ideals stabilizes after finitely many steps.
- Equivalently, every ideal in \(R\) is finitely generated.

Why are Noetherian rings important?
- They ensure the finiteness conditions needed for many algebraic theorems.
- They guarantee that the spectrum of the ring, \(\operatorname{Spec}(R)\), has well-behaved properties.
- Many classical rings encountered in algebraic geometry are Noetherian, such as polynomial rings over fields.

The Krull Dimension

The Krull dimension of a ring \(R\), denoted \(\dim R\), is defined as the supremum of the lengths of chains of prime ideals:

\[
\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n
\]

where each inclusion is strict, and the chain corresponds to a chain of prime ideals in $\operatorname{Spec}(R)$.

Key properties:
- The dimension reflects the "complexity" or "size" of the spectrum.
- For polynomial rings over a field, the dimension increases by the number of variables.

Prime Ideals, Localization, and Dimension

Prime Ideals and Their Significance

- Prime ideals are crucial in understanding the structure of rings.
- The set of prime ideals, $\operatorname{Spec}(R)$, forms a topological space with the Zariski topology.
- The dimension of the ring is linked to the chains of prime ideals within this space.

Localization at a Prime Ideal

- Given a prime ideal $\mathfrak{p}$, the localization $R_\mathfrak{p}$ focuses on the behavior around $\mathfrak{p}$.
- The dimension of $R_\mathfrak{p}$, denoted $\dim R_\mathfrak{p}$, measures the local complexity at $\mathfrak{p}$.

The Main Problem: Dimension of a Ring and Prime Ideals Containing a Non-zero Element

Let's analyze the specific problem:

> Let R be a Noetherian ring, and $F \in R$ be a non-zero element. Show that:

> $$
> \dim R = \max \{ \dim R / P : P \text{ is a prime ideal with } F \in P \}
> $$

Interpretation:
- The goal is to relate the dimension of R with the maximum dimension of the quotients R/P where P contains F.

Understanding the Statement

- The statement suggests that the dimension of the whole ring R can be recovered by looking at prime ideals that contain F.
- Since $F \neq 0$, it is contained in some prime ideals, and these prime ideals influence the structure of R.

Step-by-Step Proof and Explanation

Step 1: Prime Ideals Containing F and Their Role

- Every element $F \neq 0$ is contained in at least one prime ideal P.
- The set of prime ideals containing F forms a closed subset in the spectrum $\operatorname{Spec}(R)$.

Step 2: Chains of Prime Ideals and Dimension

- The Krull dimension of R is the maximum length of chains of prime ideals.
- For a prime ideal P containing F, the chain length in R that ends at P corresponds to the chain length in R/P.

Step 3: Comparing $\dim R$ and $\dim R/P$

- The dimension of R can be viewed as:

$$
\dim R = \sup_{Q \in \operatorname{Spec}(R)} \operatorname{ht}(Q)
$$

where $\operatorname{ht}(Q)$ is the height of Q.

- For a prime ideal P containing F:

$$
\dim R = \operatorname{ht}(P) + \dim R/P
$$

since the height of P plus the dimension of the quotient R/P equals the dimension of R.

Step 4: Establishing the Equality

- The maximum dimension among R/P for all P with $F \in P$ corresponds to the largest possible $\dim R/P$.
- Since:

$$
\dim R = \sup_{F \in P} (\operatorname{ht}(P) + \dim R/P)
$$

and $\operatorname{ht}(P) \ge 0$, it follows that:

$$
\dim R = \max_{F \in P} \dim R/P
$$

because the height can vary, but the maximum $\dim R/P$ over all such primes captures the entire dimension of R.

Implications and Applications

Understanding the Structure of Rings via Elements

- This result allows algebraists to analyze the overall dimension of a ring by examining quotients associated with prime ideals containing specific elements.
- It provides a method to "localize" the dimension problem to certain prime ideals, simplifying calculations.

Applications in Algebraic Geometry

- When studying algebraic varieties, the rings R often correspond to coordinate rings.
- The element F might represent a regular function, and understanding the primes containing F relates to the geometric properties of the variety, like irreducible components or subvarieties.

Further Theoretical Insights

- The result highlights the importance of prime ideals containing specific elements in understanding the ring's global structure.
- It also emphasizes the relevance of localization and quotient rings in dimension theory.

Conclusion

This exploration of Noetherian rings and their dimensions reveals a fundamental relationship between the dimension of a ring and the dimensions of its quotients by prime ideals containing a given non-zero element. The key takeaway is that the Krull dimension of \(R\) can be characterized as the maximum dimension of the quotients \(R/P\) over all prime ideals \(P\) containing \(F\). This insight is instrumental in both theoretical investigations and practical computations within algebra and algebraic geometry, offering a powerful tool for understanding the structural complexity of rings and their spectra.

References and Further Reading

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Note: This article provides a comprehensive overview suitable for students and researchers interested in the foundations of algebraic structures, particularly in the study of Noetherian rings and their dimensions.

Frequently Asked Questions

What does it mean for R to be a Noetherian ring in the context of dimension theory?

A Noetherian ring R is a ring in which every ascending chain of ideals stabilizes, meaning it satisfies the ascending chain condition. This property ensures that the spectrum of R, Spec(R), has well-behaved topological and algebraic properties, making the study of dimensions and chains of prime ideals manageable and well-structured.

How is the Krull dimension of a Noetherian ring R defined?

The Krull dimension of a Noetherian ring R is defined as the supremum of the lengths n of chains of prime ideals $P_0 \subset P_1 \subset \ldots \subset P_n$ in Spec(R), where each inclusion is strict. It measures the 'height' of the longest chain of prime ideals in R.

What is the significance of the set of prime ideals P containing a non-zero element F in R?

The set of prime ideals P containing a non-zero element F in R, denoted V(F), forms a closed subset of Spec(R). Studying the dimension of R localized at these primes helps understand the structure of R near the element F and how it influences the chain of prime ideals.

How does the dimension of R relate to the dimensions of R/P for prime ideals P?

The dimension of R equals the maximum of the dimensions of the quotient rings R/P as P varies over prime ideals in Spec(R). This relationship reflects the idea that the dimension of a ring can be understood by examining its prime spectrum and the dimensions of its residue fields.

What is the goal of showing that dim R equals the maximum of dim R/P for P in Spec(R)?

The goal is to establish that the Krull dimension of R can be determined by analyzing the dimensions of its residue rings R/P. This helps reduce problems about the entire ring to problems about simpler quotient rings, facilitating inductive and localized arguments.

How can the element F and the associated prime ideals P with F in P be used to show that dim R equals the maximum of dim R/P?

By considering the set of primes P containing F, one can analyze chains of prime ideals passing through these primes and relate their lengths to those in R/F. Utilizing localization and properties of Noetherian rings, it follows that the dimension of R is the supremum of the dimensions of R/P over all prime P containing F, thereby establishing the equality.

Additional Resources

Dimension Theory in Noetherian Rings: An Expert Analysis

Introduction to Noetherian Rings and Dimension Theory

In the vast landscape of commutative algebra, Noetherian rings occupy a central position due to their rich structural properties and their pivotal role in algebraic geometry. These rings, characterized by the ascending chain condition on ideals, serve as the algebraic backbone for understanding algebraic varieties, schemes, and more. One of the fundamental concepts associated with these rings is the notion of dimension—a measure of the "size" or "complexity" of the algebraic object in question.

When considering a Noetherian ring R, the spectrum $\operatorname{Spec}(R)$—the set of all prime ideals—becomes a topological space with the Zariski topology. The dimension of R, denoted $\dim R$, is defined as the supremum of the lengths of chains of prime ideals within this spectrum. This concept is not just an abstract measure; it deeply influences the structure of algebraic varieties and their geometric properties.

In this article, we delve into a specific aspect of the dimension theory: understanding how the dimension of a Noetherian ring relates to the dimensions of its quotients by prime ideals, especially in the presence of a fixed element $F \in R$. We will explore the statement:

> Given a Noetherian ring R and a non-zero element $F \in R$, show that
> $$
> \dim R = \max_{P \in \operatorname{Max}(R)} \left\{ \dim R/P : P \supseteq (F) \right\}
> $$
>
> and analyze the underlying ideas and proofs involved.

This investigation not only illuminates the intrinsic structure of Noetherian rings but also exemplifies key techniques in commutative algebra, such as localization, chain conditions, and dimension theory.

Understanding the Core Concepts

What Is a Noetherian Ring?

A Noetherian ring R is a ring for which every ascending chain of ideals stabilizes. Formally, if

$$
I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots
$$

is a chain of ideals in R, then there exists some n such that for all $m \geq n$, $I_m = I_n$.

This property ensures that ideals in R are well-behaved, preventing infinitely increasing chains and allowing the use of finiteness conditions in proofs. Examples include:

- Polynomial rings over fields, such as $k[x_1, \dots, x_n]$,
- The ring of integers $\mathbb{Z}$,
- Any finitely generated algebra over a Noetherian ring.

The Noetherian condition is fundamental in algebraic geometry because it guarantees that algebraic varieties are defined by finitely many equations.

Prime Ideals and the Spectrum $\operatorname{Spec}(R)$

The spectrum of a ring, $\operatorname{Spec}(R)$, is the set of all prime ideals of R. It serves as the algebraic counterpart to geometric spaces. The topology on $\operatorname{Spec}(R)$ is generated by the closed sets:

$$
V(I) = \{ P \in \operatorname{Spec}(R) : I \subseteq P \}
$$

for ideals $I \subseteq R$.

Within this space, maximal ideals $\operatorname{Max}(R)$ are of particular interest because the quotient R/M for a maximal ideal M is a field, reflecting points in the geometric picture.

Dimension Theory in Noetherian Rings

Definition of the Krull Dimension

The Krull dimension of a ring R, denoted $\dim R$, is:

$$
\dim R = \sup \left\{ n \in \mathbb{N} \cup \{\infty\} : \text{there exists a chain of prime ideals} \
P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \right\}
$$

In simpler terms, it measures the maximum length of chains of prime ideals.

For example, in the polynomial ring $k[x_1, \dots, x_n]$ over a field k, the dimension is n, reflecting the dimension of the algebraic variety $\mathbb{A}^n$.

Chains of Prime Ideals and Dimension

A chain of prime ideals:

$$
P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n
$$

has length n, which corresponds to the "layers" of the algebraic structure. The height of a prime ideal P is defined as:

$$
\operatorname{ht}(P) = \dim R_{P}
$$

where R_{P} is the localization of R at P. The height measures how "deep" P is within the spectrum.

Analyzing the Main Statement: Dimension and Maximal Ideals Containing F

The Statement Restated

Let R be a Noetherian ring, and $F \in R$ a non-zero element. The goal is to show:

$$
\dim R = \max_{P \in \operatorname{Max}(R), \, P \supseteq (F)} \left\{ \dim R/P \right\}
$$

This highlights that the dimension of R can be realized as the maximum among the dimensions of quotients R/P, where P runs over the maximal ideals containing the element F.

Step-by-Step Demonstration and Theoretical Justification

Step 1: Understanding the Role of F in the Spectrum

Since F is a non-zero element, it is contained in some prime ideals. The set of prime ideals containing (F):

$$
V(F) = \{ P \in \operatorname{Spec}(R) : F \in P \}
$$

is a closed subset of $\operatorname{Spec}(R)$. Within this set, the maximal ideals containing \(

(F) \) are of particular interest because the quotient \(R/P \) is a field, and the dimension of \(R/P \) is zero (since fields have Krull dimension zero).

However, the statement emphasizes the relationship between the dimension of \(R \) and the dimensions of certain quotients by prime ideals containing \(F \). This suggests that the "largest" possible dimension of \(R/P \) for such \(P \) reflects the overall dimension of \(R \).

Step 2: Utilizing the Dimension of Quotients and Chains of Prime Ideals

Recall that:

\[
\dim R = \sup \{ \operatorname{ht}(P) + \dim R/P : P \in \operatorname{Spec}(R) \}
\]

Since \(R \) is Noetherian, Krull's principal ideal theorem and related results give us tools to analyze the heights and dimensions.

Observe that for a prime ideal \(P \supseteq (F) \), the quotient \(R/P \) has dimension:

\[
\dim R/P
\]

which can be viewed as the dimension of the residual space after "factoring out" \(P \).

The key idea is that within \(V(F) \), the prime ideals \(P \) with maximal dimension of \(R/P \) are those that contribute to the overall dimension of \(R \).

Step 3: Connecting the Maximal Dimension to Prime Ideals Containing \(F \)

The statement can be understood via the following argument:

- The dimension of \(R \) is the supremum of the lengths of chains of prime ideals.
- Any chain achieving \(\dim R \) must include primes \(P \) with properties related to \(F \), especially when looking at the spectrum subset \(V(F) \).

More precisely, the chain:

\[

0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n
\]

can be "refined" to include a prime \(P \) containing \((F) \), such that the dimension of the quotient \(R/P \) reflects the residual complexity.

Formal Proof Sketch

1. Upper bound:

For any maximal ideal \(P \supseteq (F) \):

\[
\dim R \geq \dim R/P
\]

because the dimension of \(R \) is at least as large as the dimension of any of its quotients.

2. Existence of primes \(P \supseteq (F) \) realizing the maximum:

Since \(R \) is Noetherian, the set \(V(F) \) is closed, and

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