Let R Be A Relation On The Set Of All Integers Such That ARb If And Only If 3a - 5b Is Even. 1) Is R

Let R Be A Relation On The Set Of All Integers Such That ARb If And Only If 3a - 5b Is Even. 1) Is R

Understanding the nature of relations on sets, especially on the set of all integers, is fundamental in abstract algebra and discrete mathematics. In particular, relations defined by algebraic conditions often exhibit interesting properties such as reflexivity, symmetry, transitivity, and equivalence. This article explores the relation R on the set of all integers, where for any integers a and b, the relation ARb holds if and only if 3a - 5b is even. We will analyze whether R is an equivalence relation, examine its properties, and understand its significance in mathematical contexts.

What Is The Relation R?

Definition of R

The relation R is defined on the set of all integers, $\mathbb{Z}$, such that for any a, b $\in \mathbb{Z}$,

- ARb if and only if 3a - 5b is an even number.

This means that the pair (a, b) belongs to R precisely when the difference 3a - 5b is divisible by 2, i.e., when it is an even integer.

Initial Intuition

At first glance, R relates two integers based on a simple parity condition involving linear combinations of a and b. Since parity (evenness or oddness) plays a central role, understanding how multiples of integers influence parity is crucial in analyzing R's properties.

Analyzing the Properties of R

To determine whether R qualifies as an equivalence relation, we need to analyze three key properties: reflexivity, symmetry, and transitivity.

Reflexivity of R

A relation R is reflexive if for every $a \in \mathbb{Z}$, $(a, a) \in R$.

Let's check whether R is reflexive:

- For a given integer a, consider $3a - 5a = (3 - 5)a = -2a$.
- Since $-2a$ is always even (as any integer multiplied by 2 yields an even number), $3a - 5a$ is even.
- Therefore, $(a, a) \in R$ for all $a \in \mathbb{Z}$.

Conclusion: R is reflexive.

Symmetry of R

A relation R is symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$.

Let's analyze:

- Suppose $(a, b) \in R$, so $3a - 5b$ is even.
- We need to check whether this implies $3b - 5a$ is even as well.

Expressed differently:

- Given that $3a - 5b \equiv 0 \mod 2$,
- Is $3b - 5a \equiv 0 \mod 2$?

Let's analyze the parity:

- $3a - 5b \equiv 0 \mod 2$
- Since 3 and 5 are odd, their parity influences the entire expression.

Note:

- $3a \equiv a \mod 2$ (because $3 \equiv 1 \mod 2$),
- $5b \equiv b \mod 2$.

Therefore:

- $3a - 5b \equiv a - b \mod 2$.

Similarly,

- $3b - 5a \equiv b - a \mod 2$.

But $a - b \equiv -(b - a) \mod 2$, so:

- $3a - 5b \equiv a - b \mod 2$,
- $3b - 5a \equiv b - a \equiv -(a - b) \mod 2$.

Since in mod 2, negation doesn't change the parity:

- $a - b \equiv -(a - b) \bmod 2$.

Thus, if $a - b \equiv 0 \bmod 2$, then $b - a \equiv 0 \bmod 2$, and vice versa.

Conclusion: R is symmetric because the parity condition depends on a - b, which is symmetric in this sense.

Transitivity of R

A relation R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Let's analyze:

- Assume $(a, b) \in R$ and $(b, c) \in R$.

- That means:

1) 3a - 5b is even.

2) 3b - 5c is even.

- We need to check whether 3a - 5c is even.

Express:

- $3a - 5c = (3a - 5b) + (5b - 5c) = (3a - 5b) + 5(b - c)$.

- Since 3a - 5b is even, and $5(b - c) = 5 (b - c)$.

- Note that 5 is odd, so the parity of 5(b - c) depends solely on (b - c).

- Because 3b - 5c is even, and as shown, this implies that a relation involving b - c's parity exists.

Let's analyze the parity of (b - c):

- $3b - 5c \equiv b - c \bmod 2$ (as shown earlier).

- Since 3b - 5c is even, then $b - c \equiv 0 \bmod 2$.

- Now, considering 5(b - c): since 5 is odd and b - c is even, their product is even (odd even = even).

- Therefore, 5(b - c) is even.

- Summing:

- (3a - 5b) (even) + 5(b - c) (even) = even + even = even.

- So, 3a - 5c is even.

Conclusion: R is transitive.

Overall, since R is reflexive, symmetric, and transitive, R is an equivalence relation on the set of integers.

Is R An Equivalence Relation?

Given the above properties:

- Reflexive: Yes, as shown, for all a in $\mathbb{Z}$, 3a - 5a = -2a, which is always even.

- Symmetric: Yes, because the condition reduces to parity of a - b, which is symmetric.

- Transitive: Yes, as demonstrated through algebraic reasoning involving parity.

Therefore, R is an equivalence relation on the set of all integers.

Understanding the Equivalence Classes of R

Since R is an equivalence relation, it partitions the set of integers into distinct equivalence classes. Let's characterize these classes.

Characterizing the Classes

Recall that $(a, b) \in R$ if and only if 3a - 5b is even.

From previous analysis:

- $3a - 5b \equiv a - b \mod 2$ (since $3a \equiv a \mod 2$ and $5b \equiv b \mod 2$).

- Therefore, $(a, b) \in R$ if and only if $a - b \equiv 0 \mod 2$.

- This simplifies to: a and b have the same parity.

Implication:

- The equivalence classes are determined by the parity of integers.

- All integers with even parity form one class, and all odd integers form another.

Conclusion:

- The set of all integers $\mathbb{Z}$ splits into two equivalence classes:

1) Even integers: { ..., -4, -2, 0, 2, 4, ... }

2) Odd integers: { ..., -3, -1, 1, 3, 5, ... }

- Two integers are related if and only if they share the same parity.

Applications and Significance of R

Understanding this relation has several mathematical and computational implications.

Partitioning of Integers

The relation R partitions the set of integers into two simple classes based on parity. This is fundamental in number theory, especially in problems involving parity, divisibility, and modular arithmetic.

Modeling Parity in Algorithms

In computer science, parity checks are essential in error detection algorithms, data encoding, and cryptography. Relations like R help formalize these concepts mathematically.

Algebraic Structures

The set of integers under this relation forms two equivalence classes, which are fundamental in constructing quotient sets and understanding modular arithmetic.

Summary: Is R an Equivalence Relation?

To summarize:

- The relation R is defined on the set of all integers such that ARb if and only if 3a - 5b is even.

- R is an equivalence relation because it is:

- Reflexive (every integer relates to itself).

- Symmetric (parity depends on the difference a - b).

- Transitive (parity of differences propagates through chains).

- The equivalence classes are characterized by the parity of integers:

- All even integers belong to one class.

- All odd integers belong to another.

- This relation neatly partitions ℤ

Frequently Asked Questions

Is the relation R on the set of all integers reflexive?

Yes, R is reflexive because for any integer a, 3a - 5a = -2a, which is always even since -2a is divisible by 2.

Is the relation R symmetric?

Yes, R is symmetric because if 3a - 5b is even, then 3b - 5a is also even, ensuring the relation holds both ways.

Is the relation R transitive?

Yes, R is transitive. If 3a - 5b and 3b - 5c are even, then their sum and differences can be combined to show 3a - 5c is also even.

Can the relation R be classified as an equivalence relation?

Yes, since R is reflexive, symmetric, and transitive, it qualifies as an equivalence relation on the set of integers.

What is the equivalence class of an integer a under relation R?

The equivalence class of a consists of all integers b such that 3a - 5b is even, which can be characterized by certain congruence conditions modulo 2.

How can we describe the relation R in terms of modular arithmetic?

The relation R holds between a and b if and only if $3a - 5b \equiv 0 \pmod{2}$, meaning 3a and 5b have the same parity.

Additional Resources

Let R Be A Relation On The Set Of All Integers Such That ARb If And Only If 3a - 5b Is Even is a fascinating topic that explores the properties of relations defined over the set of integers. This relation, denoted as R, is intriguing because it involves a simple algebraic condition—namely, the parity (evenness or oddness) of a linear combination of integers. Analyzing whether R exhibits properties such as reflexivity, symmetry, transitivity, and whether it qualifies as an equivalence relation or a partial order provides insight into the structure and behavior of such relations. This article aims to dissect the relation R thoroughly, exploring its fundamental features, and offering a comprehensive understanding of its mathematical implications.

Understanding the Relation R

Definition and Intuition

The relation R on the set of all integers, $\mathbb{Z}$, is defined as:

> For any integers a and b, a R b if and only if 3a - 5b is even.

This definition hinges on the parity of a specific linear combination. To interpret this, consider the roles of 3a and 5b:

- Since 3 and 5 are constants, the parity of 3a depends solely on the parity of a because 3 times an integer retains the parity of that integer (since 3 is odd, multiplying by an integer preserves the parity).

- Similarly, 5b, with 5 odd, also preserves the parity of b.

Therefore, the relation essentially compares the parity of a and b through the linear combination 3a - 5b.

Key observation: Because 3 and 5 are both odd, the parity of 3a is the same as the parity of a, and the parity of 5b is the same as the parity of b.

Simplification: The parity of 3a - 5b reduces to the parity of a - b because both multipliers are odd.

Characterizing the Parity Condition

Parity of 3a - 5b

Let's analyze the parity:

- Since 3a has the same parity as a (because 3 is odd),
- and 5b has the same parity as b (because 5 is odd),

we have:

> 3a - 5b is even if and only if a - b is even.

Proof:

- Parity of 3a: same as a
- Parity of 5b: same as b

- Therefore, 3a - 5b has the same parity as a - b

Hence, the relation R simplifies to:

> a R b if and only if a - b is even.

This simplification is crucial because it allows us to analyze R as a relation based solely on the parity of the difference between two integers.

Analyzing the Properties of R

Having established that R is equivalent to "a - b is even," we can now proceed to analyze its properties.

Reflexivity

- Definition: A relation R on a set is reflexive if, for every element a, a R a holds.

- Check: For any integer a, is a R a?

- Since a - a = 0, which is even, the relation holds for all a in $\mathbb{Z}$.

Conclusion: R is reflexive.

Symmetry

- Definition: R is symmetric if, whenever a R b, then b R a.

- Check: Suppose a R b, which means a - b is even.

- Then, b - a = -(a - b), which is also even (since the negation of an even number is even).

- Therefore, if a R b, then b R a.

Conclusion: R is symmetric.

Transitivity

- Definition: R is transitive if, whenever a R b and b R c, then a R c.

- Check: Suppose:

- a R b → a - b is even,
- b R c → b - c is even.

- Adding these equations:

(a - b) + (b - c) = a - c, which is the sum of two even numbers, hence even.

- Therefore, a R c, since a - c is even.

Conclusion: R is transitive.

Is R An Equivalence Relation?

Given the properties:

- Reflexive
- Symmetric
- Transitive

The relation R satisfies all the criteria for being an equivalence relation on the set of integers.

Implications:

- The set ℤ is partitioned into equivalence classes based on this relation.

Equivalence Classes of R

Since R relates two integers if their difference is even, the equivalence classes are:

- Class of even integers: All integers that differ by an even number from each other. Specifically, all even integers form one class.

- Class of odd integers: All integers that differ by an even number from each other, but are odd. All odd integers form another class.

Explicitly:

- [even]: The set of all even integers, i.e., {..., -4, -2, 0, 2, 4, ...}
- [odd]: The set of all odd integers, i.e., {..., -3, -1, 1, 3, ...}

These two classes partition the set ℤ into two disjoint subsets.

Summary of Features of R

Feature	Description	Result
Reflexivity	For all a, a R a	Yes
Symmetry	If a R b, then b R a	Yes
Transitivity	If a R b and b R c, then a R c	Yes
Equivalence relation	Satisfies reflexivity, symmetry, transitivity	Yes
Equivalence classes	Partition into even and odd integers	Yes

Pros:

- Simple and easy to verify properties based on parity.
- Partitions $\mathbb{Z}$ into just two classes, making it straightforward to analyze.
- Useful in studying parity-based properties and modular arithmetic.

Cons:

- Limited to parity distinctions; does not distinguish between individual integers within classes.
- Not a partial order, as it's symmetric (not antisymmetric).
- Its utility is confined to parity-based relations; lacks finer granularity.

Additional Considerations and Variations

Relation R with Different Conditions

While the current relation is based on the parity of 3a - 5b, similar relations could be considered by changing coefficients or conditions:

- For example, a R' b if 2a + 3b is divisible by some integer.
- Or relations based on modular congruences, such as $a \equiv b \mod n$.

These variations can lead to different classes and properties, enriching the study of relations on integers.

Applications of R

- Parity-based relations are fundamental in number theory.
- Useful in solving equations where parity constraints are involved.

- Can serve as a foundational concept for modular arithmetic and equivalence classes modulo 2.

Conclusion

In conclusion, the relation R defined on the set of integers by the condition that 3a - 5b is even simplifies to a relation based on the parity of the difference between a and b. This relation exhibits all the key properties of an equivalence relation—reflexivity, symmetry, and transitivity—and partitions the integers into two distinct classes: even and odd integers. Its simplicity and clear structure make it an excellent example of equivalence relations in number theory and discrete mathematics. Understanding such relations enhances our grasp of modular arithmetic and the foundational concepts of set partitioning, equivalence classes, and parity.

Final note: Relations like R demonstrate how algebraic conditions translate into fundamental properties that structure the entire set, revealing the inherent symmetry and organization within the integers.

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