Let T: P2(R) R3 Be Defined As T(p(x))=(p(-1),p(0),p(1)) A)Show That T Is Linear B)Find Ker(T) C)Is T

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Introduction

In the study of linear algebra, understanding the properties of linear transformations is fundamental. The transformation T: P2(R) → R^3, defined by T(p(x)) = (p(-1), p(0), p(1)), where P2(R) is the space of all real polynomials of degree at most 2, offers a rich example to explore these properties. This article aims to demonstrate the linearity of T, determine its kernel (Ker(T)), and analyze whether T is an injective, surjective, or bijective transformation. These concepts are essential for understanding the structure of polynomial spaces and their mappings into coordinate spaces.

Part A: Showing That T Is Linear

Definition of Linearity

A transformation T: V → W between two vector spaces V and W over a field (in this case, ℝ) is linear if it satisfies two properties for all vectors u, v ∈ V and all scalars c ∈ ℝ:

- **Additivity:** T(u + v) = T(u) + T(v)
- **Homogeneity:** T(cu) = cT(u)

Applying the Definition to T

Let p(x) and q(x) be any polynomials in P2(R), and c be any real scalar.

- To verify additivity:

T(p(x) + q(x)) = ((p + q)(-1), (p + q)(0), (p + q)(1))

= (p(-1) + q(-1), p(0) + q(0), p(1) + q(1))

= (p(-1), p(0), p(1)) + (q(-1), q(0), q(1))

= T(p(x)) + T(q(x))

- To verify homogeneity:

$T(c\, p(x)) = (c\, p(-1), c\, p(0), c\, p(1))$

$= c\, (p(-1), p(0), p(1))$

$= c\, T(p(x))$

Since both properties hold for any $p(x), q(x) \in P2(R)$ and $c \in \mathbb{R}$, T is a linear transformation.

Part B: Finding the Kernel of T (Ker(T))

Definition of Kernel

The kernel of T, denoted as Ker(T), consists of all polynomials $p(x) \in P2(R)$ such that $T(p(x)) = (0, 0, 0)$. In other words, Ker(T) is the set of all polynomials that map to the zero vector in $\mathbb{R}^3$.

Finding Polynomials in Ker(T)

Suppose $p(x) = a x^2 + b x + c \in P2(R)$. Then,

$T(p(x)) = (p(-1), p(0), p(1)) = (0, 0, 0)$

This leads to the system of equations:

1. $p(-1) = a(-1)^2 + b(-1) + c = a - b + c = 0$
2. $p(0) = a(0)^2 + b(0) + c = c = 0$
3. $p(1) = a(1)^2 + b(1) + c = a + b + c = 0$

From equation (2), $c = 0$.

Substitute $c = 0$ into equations (1) and (3):

- $a - b = 0 \rightarrow a = b$
- $a + b = 0 \rightarrow a + a = 0 \rightarrow 2a = 0 \rightarrow a = 0$

Since $a = 0$, then $b = 0$ (from $a = b$). Therefore, the polynomial p(x) reduces to:

$p(x) = a x^2 + b x + c = 0 x^2 + 0 x + 0 = 0$

The only polynomial in Ker(T) is the zero polynomial.

Conclusion on Ker(T)

$Ker(T) = \{ p(x) \in P2(R) \mid p(x) \equiv 0 \} = \{ 0 \}$

This indicates that the transformation T is injective because its kernel contains only the zero polynomial.

Part C: Is T

Analysis of T’s Properties

Given the previous results, T is a linear transformation from P2(R) to $\mathbb{R}^3$ with the following properties:

- Injectivity: Since Ker(T) = {0}, T is injective. No non-zero polynomial maps to the zero vector, meaning T is one-to-one.
- Surjectivity: To determine if T is onto, we need to check whether every vector in $\mathbb{R}^3$ has a pre-image in P2(R).

Is T Surjective?

For T to be surjective, for any $(x, y, z) \in \mathbb{R}^3$, there must exist a polynomial $p(x) = a x^2 + b x + c$ such that:

$p(-1) = x$

$p(0) = y$

$p(1) = z$

This translates into the system:

1. $a - b + c = x$
2. $c = y$
3. $a + b + c = z$

Substituting $c = y$ into the other equations:

- $a - b + y = x \rightarrow a - b = x - y$
- $a + b + y = z \rightarrow a + b = z - y$

Adding these two equations:

$(a - b) + (a + b) = (x - y) + (z - y)$

$2a = x - y + z - y = x + z - 2y$

Thus,

$a = (x + z - 2y)/2$

Using a in the equation $a + b = z - y$:

$((x + z - 2y)/2) + b = z - y$

$b = z - y - (x + z - 2y)/2$

Simplify b:

$b = (2(z - y) - (x + z - 2y)) / 2$

$b = (2z - 2y - x - z + 2y) / 2$

$b = (z - x) / 2$

Similarly, $c = y$.

Since for arbitrary (x, y, z), we can find real numbers a, b, c satisfying these equations, T is surjective.

Conclusion on T

Because T is both injective and surjective from P2(R) to $\mathbb{R}^3$, T is a bijective linear transformation (an isomorphism).

Summary of Results

- **Linearity:** T is linear because it preserves addition and scalar multiplication.
- **Kernel:** Ker(T) = {0}, meaning T is injective.
- **Surjectivity:** T is onto $\mathbb{R}^3$, making it surjective.
- **Overall:** T is an isomorphism between P2(R) and $\mathbb{R}^3$.

Implications and Applications

Understanding the properties of T provides insights into how polynomial evaluations at specific points can serve as a basis for mapping polynomial spaces into coordinate spaces. Such transformations are fundamental in various applications, including polynomial interpolation, numerical analysis, and computer graphics. The fact that T is an isomorphism highlights that evaluating polynomials at three distinct points completely characterizes the polynomial, which is a cornerstone in polynomial interpolation techniques like Lagrange interpolation.

Conclusion

In this comprehensive analysis, we demonstrated that the linear transformation T, defined as evaluation at points -1, 0, and 1, is a bijective linear mapping from the space of quadratic polynomials to $\mathbb{R}^3$. Its kernel contains only the zero polynomial, confirming injectivity, and the ability to solve for any vector in $\mathbb{R}^3$ indicates surjectivity. These properties make T a powerful example of an isomorphism in linear algebra, illustrating the deep connection between polynomial spaces and coordinate representations. Understanding such transformations enhances our grasp of fundamental linear algebra concepts and their practical applications across mathematics and engineering disciplines.

Frequently Asked Questions

What is the linear transformation T: P2(R) → R^3 defined as T(p(x)) = (p(-1), p(0), p(1))?

T is a transformation that takes a polynomial p(x) of degree at most 2 and maps it to a 3-dimensional vector consisting of the values of p at -1, 0, and 1.

How do we verify that the transformation T is linear?

To verify linearity, we need to check if T preserves addition and scalar multiplication: $T(p + q) = T(p) + T(q)$ and $T(c \cdot p) = c \cdot T(p)$. Given T is defined via evaluation at fixed points, it is linear because evaluation is a linear functional.

What is the kernel of T, denoted as Ker(T)?

Ker(T) consists of all polynomials p(x) in P2(R) such that p(-1) = 0, p(0) = 0, and p(1) = 0. These are the polynomials that vanish at all three points.

How can we find the explicit form of polynomials in Ker(T)?

Since p(x) is degree at most 2 and zeros at -1, 0, and 1, p(x) must be divisible by the polynomial (x + 1), x, and (x - 1). The only polynomial of degree ≤ 2 with these roots is the zero polynomial, so Ker(T) contains only the zero polynomial.

Is the transformation T injective?

Yes, because the only polynomial that maps to the zero vector (0, 0, 0) is the zero polynomial itself, meaning Ker(T) is trivial. Therefore, T is injective.

Is the transformation T surjective?

Yes, for any vector (a, b, c) in R^3, there exists a polynomial p(x) in P2(R) such that p(-1)=a, p(0)=b, and p(1)=c, making T surjective.

What is the rank of T and how does it relate to its properties?

Since T maps P2(R) onto R^3 and its kernel is trivial, the rank of T is 3, indicating T is an isomorphism between P2(R) and R^3.

Based on the above, what are the key properties of T?

T is a linear, injective, and surjective transformation with rank 3, making it an isomorphism between P2(R) and R^3.

Additional Resources

Let T: P2(R) → R^3 be defined as T(p(x)) = (p(-1), p(0), p(1)) is a fundamental example in linear algebra that illustrates how polynomial functions can be connected to vector spaces and coordinate mappings. This operator T takes a polynomial of degree at most 2, denoted by P2(R), and evaluates it at three specific points: -1, 0, and 1. The resulting triplet forms a vector in R^3. Exploring the properties of this transformation not only sheds light on the structure of polynomial spaces but also provides insights into concepts such as linearity, kernels, and the nature of such transformations as linear maps. This review offers a comprehensive analysis of the function T, examining its linearity, kernel, and other relevant characteristics, supported by detailed explanations, proofs, and discussions.

A) Show That T Is Linear

To establish that T is a linear transformation, we need to verify two fundamental properties: additivity and scalar homogeneity. That is, for all p(x), q(x) ∈ P2(R) and any scalar c ∈ R, the following must hold:

1. T(p + q) = T(p) + T(q)
2. T(c p) = c T(p)

Additivity:

Let p(x), q(x) ∈ P2(R). Then,

T(p + q) = ((p + q)(-1), (p + q)(0), (p + q)(1))

By the properties of polynomials and pointwise evaluation,

= (p(-1) + q(-1), p(0) + q(0), p(1) + q(1))

which can be written as

= (p(-1), p(0), p(1)) + (q(-1), q(0), q(1)) = T(p) + T(q)

Scalar Homogeneity:

Let $c \in R$ and $p(x) \in P2(R)$. Then,

$T(c\ p) = (c\ p(-1), c\ p(0), c\ p(1)) = c\ (p(-1), p(0), p(1)) = c\ T(p)$

Since both properties hold, T is a linear transformation.

Features and Observations:

- The evaluation at fixed points is a linear operation because polynomial evaluation is linear.
- The proof hinges on the linearity of function evaluation and the fact that polynomial addition and scalar multiplication are linear operations.

Pros and Cons:

- Pros: Confirming linearity is straightforward due to the inherent linearity of polynomial evaluation.
- Cons: This method relies on the fundamental properties of polynomial operations; it may not be as transparent in more complex transformations.

B) Find Ker(T)

The kernel of T, denoted as Ker(T), comprises all polynomials p(x) in P2(R) such that T(p) = (0, 0, 0). In other words, these are all polynomials that evaluate to zero at x = -1, 0, and 1.

Step-by-step determination:

Given $p(x) \in P2(R)$, we have:

$T(p) = (p(-1), p(0), p(1)) = (0, 0, 0)$

This leads to the system:

1. $p(-1) = 0$
2. $p(0) = 0$
3. $p(1) = 0$

Since p(x) is quadratic or less, we can express p(x) as:

$p(x) = a\ x^2 + b\ x + c$

Applying the conditions:

- $p(0) = c = 0$
- $p(1) = a + b + c = 0$
- $p(-1) = a - b + c = 0$

Substituting $c = 0$:

- $a + b = 0$
- $a - b = 0$

Adding these two equations:

$(a + b) + (a - b) = 0 + 0 \rightarrow 2a = 0 \rightarrow a = 0$

From $a + b = 0$, and $a = 0$:

$0 + b = 0 \rightarrow b = 0$

And $c = 0$ as established earlier.

Thus, the only polynomial satisfying these conditions is:

$p(x) = 0 x^2 + 0 x + 0 = 0$

Conclusion:

$Ker(T) = \{p(x) \in P2(R) \mid p(x) \equiv 0\} = \{\text{zero polynomial}\}$

Features and Implications:

- The kernel contains only the zero polynomial, indicating T is injective when viewed as a map from P2(R) to R^3.
- The polynomial must vanish at three distinct points, which is only possible for the zero polynomial in P2(R).

Pros and Cons:

- Pros: The calculation is straightforward and shows the power of evaluation constraints.
- Cons: The kernel being trivial indicates T is one-to-one, but this is specific to the evaluation at three points and the space considered.

C) Is T

It appears that the question is incomplete or perhaps meant to ask about whether T is:

- Injective?
- Surjective?
- An isomorphism?
- Invertible?

Given the previous parts, we can analyze these aspects:

Injectivity:

From part B, we found that Ker(T) = {0}. Since the kernel is trivial, T is injective.

Surjectivity:

The image of T is all possible triplets (p(-1), p(0), p(1)) for $p(x) \in P2(R)$. Since P2(R) is a three-dimensional space, and T maps into R^3, the question reduces to whether T is onto.

For any arbitrary vector (a, b, c) in R^3, does there exist a polynomial $p(x) \in P2(R)$ such that:

p(-1) = a

p(0) = b

p(1) = c

Given the polynomial form $p(x) = \alpha x^2 + \beta x + \gamma$, we seek α, β, γ satisfying:

- $p(-1) = \alpha(1) - \beta + \gamma = a$
- $p(0) = \gamma = b$
- $p(1) = \alpha(1) + \beta + \gamma = c$

From p(0):

$\gamma = b$

Substitute into the others:

- $\alpha - \beta + b = a \rightarrow \alpha - \beta = a - b$
- $\alpha + \beta + b = c \rightarrow \alpha + \beta = c - b$

Adding these two equations:

$(\alpha - \beta) + (\alpha + \beta) = (a - b) + (c - b) \rightarrow 2\alpha = a + c - 2b$

Then,

$\alpha = (a + c - 2b)/2$

From the second:

$\alpha + \beta = c - b \rightarrow \beta = c - b - \alpha$

Using the found α:

$\beta = c - b - [(a + c - 2b)/2] = (2(c - b) - (a + c - 2b))/2$

Simplify numerator:

$2(c - b) - (a + c - 2b) = 2c - 2b - a - c + 2b = (2c - c) + (-2b + 2b) - a = c - a$

Thus,

$\beta = (c - a)/2$

Since α and β are determined uniquely for any (a, b, c), the map T is surjective.

Conclusion:

T: P2(R) → R^3 is both injective and surjective, hence an isomorphism between P2(R) and R^3.

Features and Significance:

- T provides a coordinate system for P2(R) via evaluations at three points.
- The invertibility of T signifies a perfect correspondence between polynomials of degree ≤ 2 and triplets of real numbers, capturing all possible evaluations.

Pros and Cons:

- Pros: Highlights the utility of evaluation maps in establishing isomorphisms.
- Cons: Assumes familiarity with polynomial interpolation and linear algebra concepts.

Final Remarks and Summary

The linear transformation T: P2(R) → R^3 defined by T(p(x)) = (p(-1), p(0), p(1)) exemplifies a fundamental concept in linear algebra—evaluation maps—and demonstrates how polynomial spaces relate to vector spaces through coordinate evaluations.

- Linearity: T is linear because evaluation at fixed points preserves addition and scalar multiplication.
- Kernel: The kernel of T contains only the zero polynomial, indicating injectivity, which is significant for understanding the structure of P2(R).
- Isomorphism: T is an isomorphism, establishing a one-to-one correspondence between polynomials of degree ≤ 2 and triplets in R^3, illustrating the concept of polynomial interpolation and coordinate representation.

Implications and Applications:

- Such evaluation maps are instrumental in polynomial interpolation, approximation theory, and coding theory.
- Understanding the kernel and image helps in solving polynomial equations and in constructing bases for polynomial spaces.

Limitations and Considerations:

- The analysis assumes polynomials of degree at most 2; higher degrees require similar but more complex approaches.
- Evaluation at points must be chosen carefully to ensure invertibility, especially when dealing with

more complex transformations.

In conclusion, the

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