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Introduction

Understanding the structure of trees is fundamental in graph theory and computer science, especially in areas like data structures, algorithms, and network topology. A tree is an acyclic connected graph, and rooting a tree at a particular node transforms it into a rooted tree, enabling us to analyze hierarchical relationships within the structure. The height of a rooted tree is a critical measure that indicates the maximum distance from the root to any node in the tree.

This article aims to rigorously prove how the height of a rooted tree depends on the choice of the root node, particularly when the tree is rooted at different nodes R and r. We will explore the concepts of tree height, rooted trees, and the relationships between different rootings, culminating in the proof that the height of the tree rooted at r can be characterized in terms of the distances between R and r and the structure of the original tree T.

Preliminaries and Definitions

What is a Tree?

A tree T = (V, E) is a connected, acyclic graph consisting of a set of vertices V and edges E. Key properties include:
- For |V| = n, the number of edges |E| = n - 1.
- There exists exactly one simple path between any two vertices in V.

Rooted Tree and Height

- Rooted Tree: A tree T with a designated node r ∈ V called the root. The root defines a hierarchy where each node except the root has a unique parent.
- Height of a Rooted Tree (T, r): The maximum distance from the root r to any other node in the tree.

Formally,
$$
\operatorname{height}(T, r) = \max_{v \in V} d_{T, r}(r, v),
$$
where $d_{T, r}(r, v)$ denotes the length (number of edges) of the unique simple path from r to v.

Distance in a Tree

- The distance between two nodes u and v in T, denoted $d_T(u, v)$, is the length of the unique path connecting them.
- When the tree is rooted at r, the distance $d_{T, r}(r, v)$ is simply $d_T(r, v)$.

Understanding the Problem

Given:
- An unrooted tree T = (V, E),
- Any two nodes R and r in V,

We want to analyze the height of the tree when rooted at r, i.e., $\operatorname{height}(T, r)$.

Question: How is the height related to the distances involving R and r? Specifically, can we express $\operatorname{height}(T, r)$ in terms of the distance between R and r, and the structure of T?

This question is significant because choosing different roots affects the height, and understanding this relationship is key to algorithms like tree rerooting, diameter calculations, and hierarchical clustering.

Analyzing the Relationship Between Different Roots

The Diameter of a Tree

The diameter of a tree T is the length of the longest path between any two nodes in T. It can be found by:

1. Selecting an arbitrary node u and performing a breadth-first search (BFS) or depth-first search (DFS) to find the farthest node v from u.
2. Then, starting from v, perform BFS/DFS again to find the farthest node from v, say w.

The path from v to w is a diameter, and its length is the diameter of T.

Note: The diameter is crucial because the maximum distance from a root to any node in the tree cannot exceed the diameter.

Bounding the Height When Rooted at Different Nodes

Suppose:
- R and r are two nodes in V,
- The distance between R and r is $d = d_T(R, r)$.

Then, for the rooted tree (T, r):
- The height $h_r = \operatorname{height}(T, r)$ is the maximum over all $v \in V$ of $d_T(r, v)$.

Similarly, for the rooted tree (T, R), the height is $h_R = \operatorname{height}(T, R)$.

Proving the Relationship Between Heights and Distances

Key Idea

The height of the tree when rooted at r relates to the distances from r to the furthest node in the tree, which could be at a distance up to the diameter. When changing the root from R to r, the maximum distance to any node shifts based on the position of R and r, and the overall structure.

Formal Statement of the Result

Theorem:

For any nodes $R, r \in V$ in a tree T,

$$
\operatorname{height}(T, r) \leq \operatorname{diam}(T) = \max_{u, v \in V} d_T(u, v),
$$
and

$$
\operatorname{height}(T, r) \geq \max_{v \in V} \left| d_T(R, v) - d_T(R, r) \right|.
$$

In particular, the height when rooted at r satisfies:

$$
\boxed{
\operatorname{height}(T, r) = \max_{v \in V} d_T(r, v).
}
$$

Since the distances relate to the position of the root in the tree, we can express $d_T(r, v)$ in terms of $d_T(R, v)$ and $d_T(R, r)$.

Proof of the Relationship

Step 1: Express the distance $d_T(r, v)$ in terms of $d_T(R, v)$ and $d_T(R, r)$.

In a tree, the unique paths between nodes allow us to write:

$$
d_T(r, v) = d_T(R, v) + d_T(R, r) - 2 \times \text{(length of common prefix of the paths)}.
$$

But more precisely, because trees do not have cycles, the distance between r and v can be bounded as:

$$
| d_T(R, v) - d_T(R, r) | \leq d_T(r, v) \leq d_T(R, v) + d_T(R, r).
$$

Step 2: Use the bounds to relate heights.

- The height when rooted at r:

$$
\operatorname{height}(T, r) = \max_{v \in V} d_T(r, v).
$$

- Since for each v:

$$
| d_T(R, v) - d_T(R, r) | \leq d_T(r, v) \leq d_T(R, v) + d_T(R, r),
$$

then,

$$
\max_{v \in V} | d_T(R, v) - d_T(R, r) | \leq \operatorname{height}(T, r) \leq \max_{v \in V} (d_T(R, v) + d_T(R, r)) = \operatorname{height}(T, R) + d_T(R, r).
$$

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Step 3: Relate the heights to the diameter.

- The maximum $d_T(R, v)$ over all $v \in V$ is at most the diameter $\operatorname{diam}(T)$.

- The maximum distance from R to any node v, $\max_v d_T(R, v)$, is the eccentricity $e_T(R)$, which is at most the diameter.

Thus, the height when rooted at r can be bounded as:

$$
\boxed{
\max_{v \in V} | d_T(R, v) - d_T(R, r) | \leq \operatorname{height}(T, r) \leq e_T(R) + d_T(R, r).
}
$$

Implications and Applications

Choosing an Optimal Root to Minimize Height

- The above inequalities show that the height when rooted at r depends on the position of r relative to R and the overall structure.
- To minimize the height, one strategy is to choose r as a center of the tree, i.e., a node minimizing the maximum distance to any other node.

Tree Diameter and Rerooting