Let W Be A Random Variable Giving The Number Of Heads Minus The Number Of Tails In Three Tosses Of A

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Understanding the behavior of random variables in probability theory is fundamental to grasping more complex statistical concepts. In particular, analyzing the outcomes of coin tosses provides an excellent foundation for exploring discrete probability distributions, expected values, variances, and the properties of binomial and related distributions. This article focuses on the random variable W, which represents the difference between the number of heads and tails obtained in three tosses of a fair coin. We will dissect the probability distribution of W, examine its properties, and explore various implications for probability theory and real-world applications.

Defining the Random Variable W

What is W?

In the context of three coin tosses, each toss results in either a head (H) or a tail (T). The random variable W is defined as:

$$
W = \text{Number of Heads} - \text{Number of Tails}
$$

Given that the total number of tosses is three, the possible values of W are integers ranging from -3 to 3:

- $W = 3$ when all tosses are heads (HHH)
- $W = 2$ when there are two heads and one tail (e.g., HHT, HTH, THH)
- $W = 1$ when there is one head and two tails (e.g., HTT, THT, TTH)
- $W = 0$ when there is an equal number of heads and tails (e.g., $H T H, T H T$)
- $W = -1$ when there are two tails and one head
- $W = -2$ when there is one head and two tails
- $W = -3$ when all tosses are tails (TTT)

Note that the symmetry of the problem (fair coin) means that the probabilities for positive and negative values are mirrored.

Probability Distribution of W

Calculating Probabilities for Each Value of W

To find the probability distribution for W, we analyze the outcomes of all possible sequences of three coin tosses. Since each toss has two outcomes, the total number of outcomes is:

$$
2^3 = 8
$$

The possible outcomes and their corresponding values of W are:

Outcome	Number of Heads	Number of Tails	W
HHH	3	0	3
HHT	2	1	1
HTH	2	1	1
THH	2	1	1
HTT	1	2	-1
THT	1	2	-1
TTH	1	2	-1
TTT	0	3	-3

From this, the probabilities for each value of W are:

- $P(W=3) = \text{Probability of all heads} = \frac{1}{8}$
- $P(W=2) = 0$ (no outcome with exactly 2 heads and 1 tail results in $W=2$)
- $P(W=1) = \text{Number of outcomes with 2 heads} / 8 = 3/8$
- $P(W=0) = 0$ (since total outcomes with equal heads and tails are only those with 1 head and 1 tail, but in three tosses, it's impossible to have exactly equal heads and tails; instead, the cases with 1 head and 2 tails or vice versa lead to $W = \pm 1$)
- $P(W=-1) = 3/8$
- $P(W=-2) = 0$
- $P(W=-3) = \frac{1}{8}$

However, note that the above table simplifies the direct counts; more precisely, the number of sequences with a given number of heads is:

- 0 heads: 1 sequence (TTT)
- 1 head: 3 sequences (HTT, THT, TTH)
- 2 heads: 3 sequences (HHT, HTH, THH)
- 3 heads: 1 sequence (HHH)

Correspondingly, the probabilities for the number of heads are:

Number of Heads	Count	Probability
0	1	$\frac{1}{8}$
1	3	$\frac{3}{8}$
2	3	$\frac{3}{8}$
3	1	$\frac{1}{8}$

Since $W = \text{Heads} - \text{Tails}$, and total tosses are 3, the relation:

$$
W = \text{Heads} - (3 - \text{Heads}) = 2 \times \text{Heads} - 3
$$

allows us to find the probabilities of each W:

Heads	$W = 2 \times \text{Heads} - 3$	Probability
0	$2 \times 0 - 3 = -3$	$\frac{1}{8}$
1	$2 \times 1 - 3 = -1$	$\frac{3}{8}$
2	$2 \times 2 - 3 = 1$	$\frac{3}{8}$

3	$2 \times 3 - 3 = 3$	$\frac{1}{8}$

Thus, the probability distribution of W is:

$$
P(W = -3) = \frac{1}{8}
$$
$$
P(W = -1) = \frac{3}{8}
$$
$$
P(W = 1) = \frac{3}{8}
$$
$$
P(W = 3) = \frac{1}{8}
$$

and the probabilities for W taking other values are zero.

Properties of the Random Variable W

Expected Value of W

The expected value, or mean, of W provides insight into its average behavior over many trials. It is calculated as:

$$
E[W] = \sum_{w} w \times P(W = w)
$$

Substituting the known probabilities:

$$
E[W] = (-3) \times \frac{1}{8} + (-1) \times \frac{3}{8} + (1) \times \frac{3}{8} + (3) \times \frac{1}{8}
$$

Calculating step-by-step:

$$
E[W] = -\frac{3}{8} - \frac{3}{8} + \frac{3}{8} + \frac{3}{8} = 0
$$

This result confirms the symmetry of the problem; on average, the difference between the number of heads and tails is zero.

Variance of W

Variance measures the spread or dispersion of the distribution of W. It is given by:

$$
Var(W) = E[W^2] - (E[W])^2

\]

Since \(E[W] = 0 \), the variance simplifies to:

\[
Var(W) = E[W^2]
\]

Calculating \(E[W^2] \):

\[
E[W^2] = (-3)^2 \times \frac{1}{8} + (-1)^2 \times \frac{3}{8} + (1)^2 \times \frac{3}{8} + (3)^2 \times \frac{1}{8}
\]

\[
E[W^2] = 9 \times \frac{1}{8} + 1 \times \frac{3}{8} + 1 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{9}{8} + \frac{3}{8} + \frac{3}{8} + \frac{9}{8} = \frac{24}{8} = 3
\]

Thus, the variance of \(W \) is:

\[
Var(W) = 3
\]

The standard deviation is the square root of the variance:

\[
\sigma_W = \sqrt{3} \approx

Frequently Asked Questions

What is the random variable W representing in the context of three coin tosses?

W represents the difference between the number of heads and tails obtained in three coin tosses.

What are the possible values that W can take after three tosses?

W can take values of -3, -1, 1, or 3, depending on the outcomes of the tosses.

How is the probability distribution of W determined for three coin tosses?

The distribution is determined by counting the number of outcomes corresponding to each value of W and dividing by total outcomes (8).

What is the probability that W equals 1 after three tosses?

The probability that W equals 1 is 3/8, corresponding to outcomes with two heads and one tail.

How many outcomes result in W being -3 in three coin tosses?

Only one outcome results in W = -3, which is when all three tosses are tails.

Can W take the value 0 after three coin tosses?

No, W cannot be 0 after three tosses because the difference between heads and tails is always odd (±1 or ±3).

What is the expected value of W in three tosses?

The expected value of W is 0 because the coin tosses are symmetric and outcomes are equally likely.

How does the symmetry of coin tosses affect the distribution of W?

Symmetry ensures that the probability of W being positive is equal to the probability of it being negative, leading to a balanced distribution centered around zero.

How can W be used to understand the outcomes of multiple coin tosses?

W provides a measure of imbalance between heads and tails, helping analyze the variability and likelihood of different outcome disparities in repeated tosses.

Additional Resources

Let W Be A Random Variable Giving The Number Of Heads Minus The Number Of Tails In Three Tosses Of A Coin—this intriguing concept opens the door to exploring fundamental ideas in probability, random variables, and combinatorics. Whether you're a student delving into introductory probability theory or a seasoned mathematician interested in the nuances of random processes, understanding how to analyze such a variable provides valuable insights into discrete probability models. This article aims to guide you through the definition, calculation, and interpretation of this particular random variable, employing clear explanations, structured methods, and illustrative examples.

Introduction to the Random Variable W

Imagine flipping a fair coin three times. Each flip results in either a head (H) or a tail (T). Now, define a random variable W as the number of heads minus the number of tails obtained in these three flips.

For example:
- If the sequence is H H T, then W = 2 (heads) - 1 (tails) = 1.
- If the sequence is T T T, then W = 0 - 3 = -3.
- If the sequence is H T H, then W = 2 - 1 = 1.

This variable captures the net "advantage" of heads over tails after three flips, offering a nuanced way to analyze the outcomes beyond just counting heads or tails independently.

Formal Definition of the Random Variable W

Let's formalize the problem:

- Sample space (S): The set of all possible outcomes from three coin flips. Since each flip has 2 outcomes, there are $2^3 = 8$ outcomes:

$$
S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}
$$

- Random variable W: For each outcome, W is defined as:

$$
W(\text{outcome}) = \text{Number of Heads} - \text{Number of Tails}
$$

Since there are three flips, the possible values of W are integers from -3 to +3, specifically:

$$
W \in \{ -3, -1, 1, 3 \}
$$

Note that W can only take odd values because the total number of flips is odd, and the difference between heads and tails must be odd as well.

Calculating the Distribution of W

To analyze W, we need to determine the probability distribution, i.e., $P(W = w)$ for each possible value of w.

Step 1: Enumerate Outcomes Corresponding to Each W

Let's list the outcomes for each value:

- W = -3: All tails

Outcomes: T T T

Number of outcomes: 1

- W = -1: One head, two tails

Outcomes: T T H, T H T, H T T

Number of outcomes: 3

- W = 1: Two heads, one tail

Outcomes: H H T, H T H, T H H

Number of outcomes: 3

- W = 3: All heads

Outcomes: H H H

Number of outcomes: 1

Step 2: Compute Probabilities

Since the coin is fair, each outcome has probability $\frac{1}{8}$:

- $P(W = -3) = \frac{1}{8}$
- $P(W = -1) = \frac{3}{8}$
- $P(W = 1) = \frac{3}{8}$
- $P(W = 3) = \frac{1}{8}$

Summary of the distribution:

W value	Outcomes	Number of outcomes	Probability
-3	T T T	1	1/8
-1	T T H, T H T, H T T	3	3/8
1	H H T, H T H, T H H	3	3/8
3	H H H	1	1/8

Visualizing the Distribution: The Probability Mass Function (PMF)

Plotting the PMF of W provides a clear visual understanding:

- The distribution is symmetric around zero, reflecting the fairness of the coin.
- Probabilities for W = ±1 are equal, as are probabilities for W = ±3.
- The maximum and minimum W values (±3) are less probable than the intermediate values (±1), consistent with the binomial distribution symmetry.

Expected Value and Variance of W

Understanding the expected value and variance of W gives insights into its average behavior and variability.

1. Expected Value $E[W]$

Given the symmetry:

$$
E[W] = \sum_{w} w \times P(W = w) = (-3) \times \frac{1}{8} + (-1) \times \frac{3}{8} + 1 \times \frac{3}{8} + 3 \times \frac{1}{8}
$$

Calculations:

$$
E[W] = -\frac{3}{8} - \frac{3}{8} + \frac{3}{8} + \frac{3}{8} = 0
$$

Interpretation: The expected net advantage of heads over tails after three flips is zero, which aligns with the fairness of the coin.

2. Variance $Var(W)$

Calculating variance:

$$
Var(W) = E[W^2] - (E[W])^2
$$

Since $E[W] = 0$:

$$
E[W^2] = \sum_{w} w^2 \times P(W=w)
$$

Compute:

$$
E[W^2] = (-3)^2 \times \frac{1}{8} + (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8}
$$

$$
E[W^2] = 9 \times \frac{1}{8} + 1 \times \frac{3}{8} + 1 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{9}{8} + \frac{3}{8} + \frac{3}{8} + \frac{9}{8} = \frac{24}{8} = 3
$$

Therefore:

$$
Var(W) = 3 - 0^2 = 3
$$

Key takeaway: The variance indicates a moderate spread around the mean, reflecting the variability in the net difference of heads and tails.

Alternative Approaches: Binomial Distribution Connection

The variable W is directly related to the binomial distribution because:

- Each coin flip is independent with probability $p = 0.5$ for heads.
- The number of heads in three flips, H, follows a Binomial distribution:

$$
H \sim \text{Binomial}(n=3, p=0.5)
$$

Since:

$$
W = H - (3 - H) = 2H - 3
$$

we can express W in terms of H:

$$
W = 2H - 3
$$

This relation simplifies the analysis:

- For $H = 0$, $W = -3$
- For $H = 1$, $W = -1$
- For $H = 2$, $W = 1$
- For $H = 3$, $W = 3$

Correspondingly, the probabilities:

$$
P(W = w) = P(H = \frac{w + 3}{2})
$$

This connection allows us to leverage binomial probabilities:

$$
P(H = k) = \binom{3}{k} \left(\frac{1}{2} \right)^3
$$

Extending the Analysis: Expectations and Distributions

1. Expected value of W via H

Using the relation $W = 2H - 3$:

$$
E[W] = 2E[H] - 3
$$

Since $E[H] = np = 3 \times 0.5 = 1.5$:

$$
E[W] = 2 \times 1.5 - 3 = 3 - 3 = 0
$$

which matches our previous calculation.

2. Variance of W

Using the variance of H:

$$
Var(H) = np(1-p) = 3 \times 0.5 \times 0.5 = 0.75
$$

Because $W = 2H - 3$, the variance scales by $2^2 = 4$:

$$
Var(W) = 4 \times Var(H) = 4 \times 0.75 = 3
$$

Again, consistent with earlier findings.

Practical Applications and Interpretations

Understanding the distribution and properties of W provides a foundation for various

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