Problem 3 Solve The Following Differential Equation Problem For X(t) Using Laplace Transforms X+2x=e

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Introduction to Differential Equations and Laplace Transforms

Differential equations are mathematical equations that relate a function to its derivatives. They are fundamental tools used in various scientific and engineering disciplines to model dynamic systems such as mechanical vibrations, electrical circuits, thermal conduction, and population dynamics. Solving these equations allows us to understand how systems evolve over time and predict future behavior.

One of the most powerful methods for solving linear ordinary differential equations (ODEs), especially those with constant coefficients, is the Laplace Transform. This integral transform converts differential equations from the time domain into an algebraic equation in the complex frequency domain, simplifying the process of finding solutions significantly.

Understanding the Problem Statement

The problem at hand is:

Solve the differential equation $X + 2x = e(t)$ for $X(t)$ using Laplace transforms.

Note: The notation in the problem suggests that $X(t)$ is the unknown function we want to solve for, and the right-hand side $e(t)$ typically represents the exponential function e^t. For clarity, the differential equation can be rewritten as:

$$
X(t) + 2 \frac{dX(t)}{dt} = e^t
$$

This is a first-order linear differential equation with constant coefficients, and solving it involves applying the Laplace transform, handling initial conditions, and then transforming back to the time domain.

Step-by-Step Solution Approach

Step 1: Recognize the Type of Differential Equation

The given equation:

$$
X(t) + 2 \frac{dX(t)}{dt} = e^t
$$

is a first-order linear ODE with constant coefficients. The standard form is:

$$
\frac{dX}{dt} + p(t)X(t) = q(t)
$$

where $p(t) = \frac{1}{2}$ and $q(t) = \frac{1}{2} e^t$, but for the Laplace transform method, we directly apply the transform to the entire differential equation.

Step 2: Apply the Laplace Transform

Recall the Laplace transform of a function $f(t)$:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt
$$

The key properties we'll use are:

- Linearity: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
- Transform of derivatives:

$$
\mathcal{L}\left\{\frac{dX(t)}{dt}\right\} = sX(s) - X(0)
$$

where $X(0)$ is the initial value of $X(t)$.

Assuming zero initial conditions (common in such problems unless specified otherwise):

$$
X(0) = 0
$$

Applying the Laplace transform to both sides:

$$
\mathcal{L}\{X(t)\} + 2 \mathcal{L}\left\{\frac{dX(t)}{dt}\right\} = \mathcal{L}\{e^t\}
$$

which becomes:

$$
X(s) + 2(sX(s) - X(0)) = \frac{1}{s - 1}
$$

Since $X(0) = 0$:

$$
X(s) + 2sX(s) = \frac{1}{s - 1}
$$

Step 3: Solve for $X(s)$

Factor $X(s)$:

$$
X(s)(1 + 2s) = \frac{1}{s - 1}
$$

Thus:

$$
X(s) = \frac{1}{(s - 1)(1 + 2s)}
$$

Step 4: Simplify the Expression

To facilitate the inverse Laplace transform, we perform partial fraction decomposition.

Rewrite:

$$
X(s) = \frac{1}{(s - 1)(2s + 1)}
$$

Note: $1 + 2s = 2s + 1$.

Set:

$$
X(s) = \frac{A}{s - 1} + \frac{B}{2s + 1}
$$

Multiply both sides by the denominator:

$$
1 = A(2s + 1) + B(s - 1)
$$

Expressed as:

$$
1 = 2A s + A + B s - B
$$

Combine like terms:

$$
1 = (2A + B)s + (A - B)
$$

Since this must hold for all s, equate coefficients:

- Coefficient of s:

$$
2A + B = 0
$$

- Constant term:

$$
A - B = 1
$$

Solve the system:

From the first:

$$
B = -2A
$$

Substitute into the second:

$$
A - (-2A) = 1 \Rightarrow A + 2A = 1 \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}
$$

Then:

$$
B = -2 \times \frac{1}{3} = -\frac{2}{3}
$$

So, the partial fractions are:

$$
X(s) = \frac{1/3}{s - 1} - \frac{2/3}{2s + 1}
$$

Rewrite the second term for easier inverse transform:

$$
X(s) = \frac{1/3}{s - 1} - \frac{2/3}{2(s + \frac{1}{2})}
$$

which simplifies to:

$$
X(s) = \frac{1/3}{s - 1} - \frac{1/3}{s + \frac{1}{2}}
$$

Inverse Laplace Transform and Final Solution

Step 5: Find the Inverse Laplace Transforms

Recall:

- $\mathcal{L}^{-1}\left\{\frac{1}{s - a}\right\} = e^{a t}$

Apply to each term:

$$
X(t) = \frac{1}{3} e^{t} - \frac{1}{3} e^{-\frac{1}{2} t}
$$

Step 6: Write the Final Solution

$$
\boxed{
X(t) = \frac{1}{3} e^{t} - \frac{1}{3} e^{-\frac{1}{2} t}
}
$$

This solution satisfies the original differential equation, assuming zero initial conditions.

Additional Considerations

Handling Initial Conditions

If initial conditions are specified (e.g., $X(0)$), they can be incorporated during the Laplace transform step, modifying the algebra accordingly.

Verifying the Solution

To confirm correctness, differentiate $X(t)$ and substitute back into the original differential equation:

$$
X(t) + 2 \frac{dX(t)}{dt} \stackrel{?}{=} e^t
$$

Calculations verify the solution's accuracy.

Practical Applications of Laplace Transforms in Differential Equations

Laplace transforms are widely used in engineering fields such as:

- Electrical engineering: Analyzing circuits with capacitors and inductors.
- Mechanical engineering: Studying vibrations and control systems.
- Control systems: Designing stable controllers.
- Signal processing: Analyzing system responses.

Their ability to convert differential equations into algebraic equations simplifies complex problems that would be cumbersome to solve directly.

Advantages of Using Laplace Transforms

- Simplifies solving linear differential equations with constant coefficients.
- Easily handles initial conditions.
- Facilitates solving equations with discontinuous or impulsive inputs.
- Provides a systematic approach applicable to a wide range of problems.

Summary

In solving the differential equation $X + 2x = e^t$ using Laplace transforms, the key steps involved:

- Recognizing the type of differential equation.
- Applying the Laplace transform considering initial conditions.
- Solving the algebraic equation for $X(s)$.
- Using partial fraction decomposition.
- Applying inverse Laplace transforms to find $X(t)$.

The resulting solution:

$$X(t) = \frac{1}{3} e^{t} - \frac{1}{3} e^{-\frac{1}{2} t}$$

provides a clear analytical expression for the system's response over time.

Conclusion

Laplace transforms serve as a powerful method for solving differential equations, especially those encountered in engineering and physics. Their ability to convert differential operations into algebraic ones simplifies the solution process, making them an essential tool for students and professionals alike. Mastery of Laplace transforms enables effective analysis and design of systems modeled by differential equations, ensuring better understanding and control of complex dynamic phenomena.

References and Further Reading

- Boyce, W. E., & DiPrima, R. C. (2017). Elementary Differential Equations and Boundary Value Problems. Wiley.
- Ogata, K. (2010

Frequently Asked Questions

What is the general form of the differential equation to be solved using Laplace transforms in Problem 3?

The differential equation is x' + 2x = e(t), where x' denotes the derivative of x with respect to t.

How do you apply Laplace transforms to solve the differential equation x' + 2x = e(t)?

Take the Laplace transform of both sides, using L{x'} = sX(s) - x(0) and L{x} = X(s), then solve for X(s).

What initial conditions are needed to solve the differential equation using Laplace transforms?

The initial value x(0) must be known; if not given, it is typically assumed to be zero for simplicity.

What is the Laplace transform of the right-hand side e(t)?

The Laplace transform of e(t), assuming e(t) = e^t, is 1/(s - 1), provided the real part of s is greater than 1.

How do you find the transformed solution X(s) after applying Laplace transforms?

Solve the algebraic equation sX(s) - x(0) + 2X(s) = 1/(s - 1) for X(s).

What is the inverse Laplace transform process to find x(t) from X(s)?

Perform partial fraction decomposition if necessary, then apply known inverse Laplace transforms to find x(t).

What is the particular solution to the differential equation in

Problem 3?

The particular solution involves the inverse Laplace transform of the algebraic expression for X(s), resulting in x(t).

How does initial condition x(0) affect the solution of the differential equation?

It influences the algebraic form of X(s) and thus the final solution x(t); zero initial conditions simplify calculations.

Can this method be generalized to solve similar linear differential equations with different right-hand sides?

Yes, Laplace transforms can be applied broadly to linear differential equations with various forcing functions, making it a versatile method.

Additional Resources

Problem 3 Solve The Following Differential Equation Problem For X(t) Using Laplace Transforms X+2x=e

In the realm of differential equations, solving for an unknown function often requires a combination of mathematical ingenuity and strategic application of transform techniques. Among these techniques, Laplace transforms stand out as a powerful tool, especially for linear differential equations with constant coefficients. Today, we delve into a classic problem: solving the differential equation \(x' + 2x = e(t) \) for \(x(t) \) using Laplace transforms. This problem not only exemplifies the elegance of the Laplace method but also underscores its practicality in engineering, physics, and applied mathematics.

Understanding the Differential Equation

Before jumping into the solution, it's crucial to understand the problem at hand. The differential equation:

\[x' + 2x = e(t) \]

is a first-order linear differential equation. Here:

- \(x' \) denotes the derivative of \(x(t) \) with respect to \(t \),
- \(2x \) indicates a proportional feedback term,
- \(e(t) \) is the input or forcing function, which, in this context, is assumed to be the exponential function \(e^t \).

This type of equation frequently models physical systems where a quantity \(x(t) \) responds to an external influence \(e^t \), such as electrical circuits with exponential sources, thermal systems, or population models.

The goal: find an explicit formula for $x(t)$ that describes the system's behavior over time.

Step 1: Recognizing the Structure and Applying the Laplace Transform

The first step in solving the differential equation involves transforming it from the time domain to the complex frequency domain using the Laplace transform. This approach simplifies differentiation into algebraic operations, making the equation easier to manipulate.

The Laplace Transform Basics:

- $\mathcal{L}\{x(t)\} = X(s)$,
- $\mathcal{L}\{x'(t)\} = sX(s) - x(0)$,

where $x(0)$ is the initial condition of the function $x(t)$. If not specified, it’s common to assume $x(0) = 0$.

Transforming the Differential Equation:

Applying the Laplace transform to both sides:

$$\mathcal{L}\{x' + 2x\} = \mathcal{L}\{e^t\}$$

becomes:

$$sX(s) - x(0) + 2X(s) = \frac{1}{s - 1}$$

since $\mathcal{L}\{e^t\} = \frac{1}{s - 1}$.

Assuming initial conditions $x(0) = 0$ simplifies the process:

$$sX(s) + 2X(s) = \frac{1}{s - 1}$$

which can be rearranged into:

$$(s + 2)X(s) = \frac{1}{s - 1}$$

Step 2: Solving for $X(s)$

To find $X(s)$, isolate it:

$$X(s) = \frac{1}{(s + 2)(s - 1)}$$

This algebraic expression is the key to obtaining $x(t)$. The next step involves decomposing $X(s)$ into simpler parts suitable for inverse Laplace transformation.

Step 3: Partial Fraction Decomposition

The expression:

$$ X(s) = \frac{1}{(s + 2)(s - 1)} $$

can be decomposed into partial fractions:

$$ \frac{1}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1} $$

To find constants A and B:

1. Multiply both sides by $(s + 2)(s - 1)$:

$$ 1 = A(s - 1) + B(s + 2) $$

2. Solve for A and B:

- Let $s = 1$:

$$ 1 = A(0) + B(3) \Rightarrow B = \frac{1}{3} $$

- Let $s = -2$:

$$ 1 = A(-3) + B(0) \Rightarrow A = -\frac{1}{3} $$

Thus,

$$ X(s) = -\frac{1}{3} \cdot \frac{1}{s + 2} + \frac{1}{3} \cdot \frac{1}{s - 1} $$

Step 4: Applying the Inverse Laplace Transform

Now, take the inverse Laplace transform of each term:

- $\mathcal{L}^{-1} \left\{ \frac{1}{s + a} \right\} = e^{-a t}$,
- $\mathcal{L}^{-1} \left\{ \frac{1}{s - b} \right\} = e^{b t}$.

Applying these:

$$ x(t) = -\frac{1}{3} e^{-2 t} + \frac{1}{3} e^{t} $$

This expression describes the response $x(t)$ of the system to the exponential input e^t.

Final Expression and Interpretation

The solution:

$$ x(t) = \frac{1}{3} e^{t} - \frac{1}{3} e^{-2 t} $$

captures both the steady-state response and the transient behavior:

- The term $\frac{1}{3} e^{t}$ signifies the system's long-term response aligned with the input e^t,
- The term $-\frac{1}{3} e^{-2 t}$ reflects the transient response that diminishes over time due to the exponential decay $e^{-2 t}$.

Initial Conditions:

If initial conditions were specified (say, $x(0) = 0$), this solution would be valid as is. For other initial states, adjustments would be necessary to account for initial values.

Broader Implications and Applications

This problem exemplifies how Laplace transforms serve as a bridge between differential equations and algebraic equations, simplifying the process of solving complex systems. The technique is especially valuable when dealing with nonhomogeneous equations, initial value problems, or systems with inputs modeled by exponential functions.

Practical applications include:

- Electrical circuit analysis, where voltage and current behaviors are modeled by differential equations,
- Mechanical systems involving damping and external forces,
- Population dynamics with exponential growth or decay factors,
- Control systems design, where stability and transient response are critical.

Conclusion: Mastering Laplace Transforms for Differential Equations

Solving differential equations like $x' + 2x = e^t$ using Laplace transforms is a fundamental skill in applied mathematics and engineering. It transforms the often intimidating calculus problem into a manageable algebraic task, providing insights into the system’s behavior both in the time and frequency domains. As demonstrated, the process involves transforming, algebraically manipulating, decomposing into partial fractions, and then returning to the time domain with inverse transforms.

Mastery of this technique not only enhances problem-solving efficiency but also deepens understanding of dynamic systems. Whether analyzing electrical circuits, mechanical vibrations, or biological systems, the principles outlined here serve as a robust foundation for tackling a wide array of differential equations with confidence.

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