Q. 6. Find The Minimum Value Of The Function F(x,y,z)= 4(x1) 2+(y2) 2+(z3) 2on The Sphere {(x,y,z):x

Q. 6. Find The Minimum Value Of The Function F(x,y,z)= 4(x1)^2 + (y2)^2 + (z3)^2 on the Sphere {(x,y,z): x^2 + y^2 + z^2 = R^2}

Introduction

Finding the minimum value of a multivariable function constrained to a surface or a region is a fundamental problem in calculus and optimization. In this context, we are asked to determine the minimum of the function:

$$F(x, y, z) = 4(x)^2 + (y)^2 + (z)^2$$

subject to the constraint:

$$x^2 + y^2 + z^2 = R^2$$

where R is a positive real number, representing the radius of the sphere.

This problem involves techniques from multivariable calculus, particularly the method of Lagrange multipliers, which is a powerful method for constrained optimization. Understanding how to approach such problems can help in various fields, including physics, engineering, and data science.

Understanding the Problem

The goal is to minimize the function F(x, y, z) over the surface of the sphere defined by $x^2 + y^2 + z^2 = R^2$. Geometrically, the problem asks: among all points on the sphere, which one yields the smallest value of the function F?

Since F involves quadratic terms, and the constraint is a sphere, the problem simplifies to analyzing the behavior of the quadratic form restricted to the sphere.

Methodology: Using Lagrange Multipliers

The method of Lagrange multipliers states that to find the extrema of a function F(x, y, z) subject to a constraint G(x, y, z) = 0, we need to solve the system:

$$\nabla F = \lambda \nabla G$$

$G(x, y, z) = 0$

where ∇F and ∇G are the gradients of F and G, respectively, and λ is the Lagrange multiplier.

In our case:

- $F(x, y, z) = 4x^2 + y^2 + z^2$
- $G(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$

Calculating the gradients:

- $\nabla F = (8x, 2y, 2z)$
- $\nabla G = (2x, 2y, 2z)$

Applying the Lagrange multiplier condition:

$(8x, 2y, 2z) = \lambda(2x, 2y, 2z)$

Component-wise equations:

1. $8x = 2\lambda x$
2. $2y = 2\lambda y$
3. $2z = 2\lambda z$

Analysis:

- For $x \neq 0$, from equation 1:

$8x = 2\lambda x \rightarrow$ dividing both sides by 2x (assuming $x \neq 0$):

$4 = \lambda$

- Similarly, for $y \neq 0$:

$2y = 2\lambda y \rightarrow \lambda = 1$

- For $z \neq 0$:

$2z = 2\lambda z \rightarrow \lambda = 1$

Since λ cannot be both 4 and 1 simultaneously unless some variables are zero, we analyze cases:

Case 1: $x \neq 0, y \neq 0, z \neq 0$

- From the equations, λ must be both 4 (from x) and 1 (from y and z), which is impossible. So, at least one of the variables must be zero.

Case 2: $x \neq 0, y = 0, z = 0$

- From the equations:

$x \neq 0 \rightarrow \lambda = 4$

- Then, check the constraint:

$x^2 + 0 + 0 = R^2 \rightarrow x = \pm R$

- The function value:

$F = 4x^2 + y^2 + z^2 = 4(R^2) + 0 + 0 = 4R^2$

Case 3: $y \neq 0$, $x = 0$, $z = 0$

- From the equations:

$y \neq 0 \rightarrow \lambda = 1$

- Constraint:

$0 + y^2 + 0 = R^2 \rightarrow y = \pm R$

- Function value:

$F = 4(0)^2 + R^2 + 0 = R^2$

Case 4: $z \neq 0$, $x = 0$, $y = 0$

- From the equations:

$z \neq 0 \rightarrow \lambda = 1$

- Constraint:

$0 + 0 + z^2 = R^2 \rightarrow z = \pm R$

- Function value:

$F = 0 + 0 + R^2 = R^2$

Case 5: Some variables are zero, others are zero or non-zero, but the previous analysis covers all possibilities for stationary points.

Determining the Minimum and Maximum Values

From the above cases, the candidate points for minima and maxima are:

- Points where $x = \pm R$, $y = 0$, $z = 0$, with $F = 4R^2$
- Points where $y = \pm R$, $x = 0$, $z = 0$, with $F = R^2$
- Points where $z = \pm R$, $x = 0$, $y = 0$, with $F = R^2$

Since the function involves squares, it is always non-negative. Comparing the values:

- $F = R^2$ (for points on the y and z axes)
- $F = 4R^2$ (for points on the x-axis)

Therefore:

- The minimum value of F on the sphere occurs at points where $y = \pm R$ or $z = \pm R$, giving $F = R^2$.
- The maximum value occurs at points where $x = \pm R$, giving $F = 4R^2$.

Summary of Results

- The minimum value of $F(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = R^2$ is R^2.
- The points where this minimum is attained are:

  $(0, \pm R, 0)$ and $(0, 0, \pm R)$

- The maximum value of F is $4 R^2$, attained at points:

  $(\pm R, 0, 0)$

Conclusion

By applying the method of Lagrange multipliers, we identified the critical points on the sphere that minimize and maximize the given quadratic function. The key takeaway is that the minimum occurs along the y and z axes, where the function reduces to R^2, while the maximum occurs along the x-axis, where it reaches $4 R^2$.

This analysis demonstrates how constrained optimization techniques efficiently solve such problems, providing critical insights into the behavior of quadratic functions on spherical surfaces. Understanding these principles is essential for advanced studies in calculus, physics, and engineering, where optimization under constraints frequently arises.

Note: When R is specified, you can substitute that value to compute the exact numerical minimum and maximum values.

Frequently Asked Questions

What is the problem asking for in the function $F(x, y, z) = 4x^2 + y^2 + z^2$ on the sphere?

It asks to find the minimum value of the function $F(x, y, z) = 4x^2 + y^2 + z^2$ subject to the constraint defined by the sphere (likely $x^2 + y^2 + z^2 = r^2$).

How do you set up the problem to find the minimum of $F(x, y, z)$ on the sphere?

Use the method of Lagrange multipliers by introducing a constraint equation (e.g., $x^2 + y^2 + z^2 = r^2$) and solving the resulting system of equations.

What is the typical constraint for the sphere in such optimization problems?

The typical constraint is $x^2 + y^2 + z^2 = R^2$, where R is the radius of the sphere.

How do Lagrange multipliers help in finding the minimum of F on the sphere?

They help by setting up equations where the gradient of F equals a scalar multiple of the gradient of the constraint, allowing us to solve for critical points that may correspond to minima or maxima.

What are the steps to find the minimum of $F(x, y, z) = 4x^2 + y^2 + z^2$ on the sphere?

1. Write the constraint equation (e.g., $x^2 + y^2 + z^2 = R^2$). 2. Set up the Lagrangian: $L = 4x^2 + y^2 + z^2 - \lambda(x^2 + y^2 + z^2 - R^2)$. 3. Find partial derivatives and solve the system of equations for x, y, z, and λ.

How do you interpret the critical points found via Lagrange multipliers in this context?

They represent points on the sphere where the function F could attain its minimum or maximum value, which can be identified by evaluations or second derivative tests.

If the sphere is defined by $x^2 + y^2 + z^2 = R^2$, what is the minimum value of F?

The minimum value occurs when the partial derivatives are zero and is typically at the point where x, y, z minimize F given the constraint; specifically, it will depend on R. If R is known, you substitute back into F to find the minimum.

Can symmetry be used to simplify the problem of minimizing F on the sphere?

Yes, symmetry considerations can reduce the complexity, especially if the function and constraint are symmetric in some variables, helping to identify candidate points for minima.

What is the significance of the coefficients in the function F(x, y, z) = 4x^2 + y^2 + z^2 regarding the minimum?

The coefficients indicate the weight or importance of each variable in the function; larger coefficients suggest that minimizing the variable associated with a higher coefficient (like x) will have a more significant impact on reducing F.

Additional Resources

Finding the Minimum Value of the Function $F(x, y, z) = 4(x^1)^2 + (y^2)^2 + (z^3)^2$ on the Sphere is a classic problem in multivariable calculus and optimization. This type of problem involves not only understanding the behavior of a complex function but also applying methods such as Lagrange multipliers to find the minimum (or maximum) values constrained to a specific surface—in this case, a sphere. Such problems are fundamental in fields ranging from physics to engineering, where optimizing a quantity under certain constraints is often required.

Understanding the Problem

Before diving into the solution, it’s crucial to carefully interpret the problem statement.

- Function to minimize:

$$
F(x, y, z) = 4(x^1)^2 + (y^2)^2 + (z^3)^2
$$

- Constraint:

$$
\text{On the sphere } \{(x, y, z) : x^2 + y^2 + z^2 = R^2 \}
$$

Typically, the sphere's radius R is given; if not specified, we may assume R is a positive constant or analyze the problem for a general radius.

Clarification of the Function

The function involves powers and coefficients:

- For the x-term: $4(x^1)^2 = 4x^2$
- For the y-term: $(y^2)^2 = y^4$
- For the z-term: $(z^3)^2 = z^6$

Therefore, the function simplifies to:

$$
F(x, y, z) = 4x^2 + y^4 + z^6
$$

Objective

Find the minimum value of $F(x, y, z)$ subject to the constraint:

$$
x^2 + y^2 + z^2 = R^2
$$

This is a constrained optimization problem. The typical approach involves:

- Using Lagrange multipliers
- Analyzing the critical points
- Evaluating the function at those points to find the minimum

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Step-by-Step Solution Approach

1. Setting Up the Lagrangian

Define the Lagrangian function:

$$
\mathcal{L}(x, y, z, \lambda) = 4x^2 + y^4 + z^6 - \lambda (x^2 + y^2 + z^2 - R^2)
$$

where λ is the Lagrange multiplier.

2. Computing the Partial Derivatives

Find the critical points by setting derivatives to zero:

$$
\frac{\partial \mathcal{L}}{\partial x} = 8x - 2\lambda x = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial y} = 4 y^3 - 2 \lambda y = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial z} = 6 z^5 - 2 \lambda z = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 + z^2 - R^2) = 0
$$

Solving the System of Equations

3. Critical Point Conditions

- For x:

$$
8x = 2 \lambda x \implies x (8 - 2 \lambda) = 0
$$

- For y:

$$
4 y^3 = 2 \lambda y \implies y (4 y^2 - 2 \lambda) = 0
$$

- For z:

$$
6 z^5 = 2 \lambda z \implies z (6 z^4 - 2 \lambda) = 0
$$

4. Possible Cases for Critical Points

The equations imply that each variable can be zero or satisfy the corresponding relations:

- Case A: $x = 0$
- Case B: $y = 0$
- Case C: $z = 0$

Alternatively, for non-zero variables:

\[

$$
8 - 2 \lambda = 0 \implies \lambda = 4
$$

$$
4 y^2 - 2 \lambda = 0 \implies 4 y^2 = 2 \lambda \implies y^2 = \frac{\lambda}{2}
$$

$$
6 z^4 = 2 \lambda \implies z^4 = \frac{\lambda}{3}
$$

Analyzing Each Case

Case 1: All variables zero (not possible on the sphere unless $R=0$) — discard since on the sphere $R > 0$.

Case 2: $x \neq 0$, $y \neq 0$, $z \neq 0$

From above:

$$
\lambda = 4
$$

$$
y^2 = \frac{\lambda}{2} = \frac{4}{2} = 2
$$

$$
z^4 = \frac{\lambda}{3} = \frac{4}{3}
$$

Now, from the sphere constraint:

$$
x^2 + y^2 + z^2 = R^2
$$

$$
x^2 + 2 + z^2 = R^2
$$

Express z^2:

$$
z^4 = \frac{4}{3} \implies z^2 = \pm \sqrt{\frac{4}{3}} \text{, but since } z^2 \ge 0, \text{ take } z^2 = \sqrt{\frac{4}{3}}
$$

But note that $z^4 = (z^2)^2$, so:

$$
z^2 = \sqrt{\frac{4}{3}} \approx 1.1547
$$

\]

Thus,

\[
x^2 = R^2 - y^2 - z^2 = R^2 - 2 - 1.1547
\]

To have real solutions:

\[
x^2 \ge 0 \implies R^2 \ge 3.1547
\]

Computing the Function Value at These Critical Points

Recall:

\[
F(x, y, z) = 4x^2 + y^4 + z^6
\]

At the critical points:

- \(y^2 = 2 \Rightarrow y^4 = (y^2)^2 = 4 \)
- \(z^4 = 4/3 \Rightarrow z^2 = \sqrt{4/3} \)

Compute \(z^6 \):

\[
z^6 = (z^2)^3 = (\sqrt{\frac{4}{3}})^3 = \left(\frac{4}{3}\right)^{3/2} = \left(\frac{4}{3}\right)^{1} \times \left(\frac{4}{3}\right)^{1/2}
\]

\[
= \frac{4}{3} \times \sqrt{\frac{4}{3}} = \frac{4}{3} \times \frac{2}{\sqrt{3}} = \frac{4 \times 2}{3 \sqrt{3}} = \frac{8}{3 \sqrt{3}}
\]

Similarly, \(4x^2 \) depends on \(x^2 \), which is:

\[
x^2 = R^2 - 2 - z^2
\]

Since \(z^2 = \sqrt{\frac{4}{3}} \), approximate:

\[
z^2 \approx 1.1547

$$

then,

$$
x^2 \approx R^2 - 2 - 1.1547 = R^2 - 3.1547
$$

Finally,

$$
F_{\text{min}} = 4x^2 + y^4 + z^6 \approx 4(R^2 - 3.1547) + 4 + \frac{8}{3 \sqrt{3}}
$$

which simplifies to:

$$
F_{\text{min}} \approx 4 R^2 - 12.6188 + 4 + 1.54 \approx 4 R^2 - 7.0788
$$

This expression indicates the minimal value depends on the sphere's radius R.

Special Cases: When Some Variables Are Zero

1. $x=0$

From the derivative condition:

$$
8x - 2 \lambda x=0 \Rightarrow 0=0
$$

No restriction here.

From the other derivatives:

$$
4 y^3 - 2 \lambda y=0 \Rightarrow y(4 y^2 - 2 \lambda)=0
$$
$$
6 z^5 - 2 \lambda z=0 \Rightarrow z(6 z^4 - 2 \lambda)=0
$$

- If $y=0$:

$$
4 y^3 -
$$

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