Question: [10] 4 2 2 Find An Orthogonal Matrix P That Diagonalises A = And Give The Matrix D= PT AP.

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Introduction to Orthogonal Matrices and Diagonalisation

Diagonalisation is a fundamental concept in linear algebra that simplifies matrix operations, especially powers and exponentials of matrices. When a matrix is diagonalised, it is expressed in a basis where it acts as a simple scaling operation, represented by a diagonal matrix. An important class of matrices that are often diagonalised are symmetric matrices, which possess real eigenvalues and orthogonal eigenvectors.

In this article, we will explore how to find an orthogonal matrix \(P \) that diagonalises a given matrix \(A \), and subsequently determine the diagonal matrix \(D = P^T A P \). The process involves understanding eigenvalues, eigenvectors, orthogonality, and the spectral theorem, which guarantees the diagonalisation of symmetric matrices via orthogonal transformations.

Understanding the Problem Statement

The question asks us to:

1. Find an orthogonal matrix \(P \) such that \(P^T A P \) is diagonal, i.e., \(D \).
2. Provide the diagonal matrix \(D \).

The key points are:

- The matrix \(A \) is given (though the specific entries are not provided in the question; we will assume a typical symmetric matrix for illustration).
- \(P \) must be orthogonal, meaning \(P^T P = I \), where \(I \) is the identity matrix.
- \(D \) will contain the eigenvalues of \(A \) along its diagonal.

Step-by-Step Process for Diagonalising a Symmetric Matrix

To systematically find the orthogonal matrix P and the diagonal matrix D, follow these steps:

1. Verify that A is symmetric

A symmetric matrix satisfies $A = A^T$. This property ensures real eigenvalues and orthogonal eigenvectors, which are necessary for orthogonal diagonalisation.

2. Find the eigenvalues of A

- Solve the characteristic equation: $\det(A - \lambda I) = 0$.
- The roots $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues.

3. Find the eigenvectors corresponding to each eigenvalue

- For each eigenvalue λ_i, solve the linear system: $(A - \lambda_i I) \mathbf{v}_i = 0$.
- Normalize these eigenvectors to unit length to ensure orthogonality.

4. Construct the orthogonal matrix P

- Arrange the normalized eigenvectors as columns in matrix P.
- Since eigenvectors are orthogonal, P will be orthogonal ($P^T P = I$).

5. Compute the diagonal matrix D

- Diagonal matrix D will have eigenvalues on its diagonal in the same order as their corresponding eigenvectors in P.

6. Confirm the diagonalisation

- Verify that $P^T A P = D$.

Illustrative Example

Let's consider a specific symmetric matrix A to demonstrate the process:

$$
A = \begin{bmatrix}
4 & 1 \\
1 & 3 \\
\end{bmatrix}
$$

Note: Since the original question does not specify A, this example will illustrate the typical approach.

Step 1: Confirm Symmetry

$$
A^T = \begin{bmatrix}
4 & 1 \\
1 & 3 \\
\end{bmatrix} = A
$$

Yes, A is symmetric.

Step 2: Find Eigenvalues

The characteristic polynomial:

$$
\det(A - \lambda I) = \det \begin{bmatrix}
4 - \lambda & 1 \\
1 & 3 - \lambda \\
\end{bmatrix} = (4 - \lambda)(3 - \lambda) - 1 \times 1
$$

$$
= (4 - \lambda)(3 - \lambda) - 1 = (12 - 4 \lambda - 3 \lambda + \lambda^2) - 1 = \lambda^2 - 7 \lambda + 11
$$

Set equal to zero:

$$
\lambda^2 - 7 \lambda + 11 = 0
$$

Solve for λ:

$$
\lambda = \frac{7 \pm \sqrt{49 - 44}}{2} = \frac{7 \pm \sqrt{5}}{2}
$$

Thus,

$$
\boxed{
\lambda_1 = \frac{7 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{7 - \sqrt{5}}{2}
}
$$

Step 3: Find Eigenvectors

For $\lambda_1 = \frac{7 + \sqrt{5}}{2}$:

Solve:

$$
(A - \lambda_1 I) \mathbf{v} = 0
$$

$$
\begin{bmatrix}
4 - \lambda_1 & 1 \\
1 & 3 - \lambda_1 \\
\end{bmatrix}
\mathbf{v} = 0
$$

Calculate entries:

$$
4 - \lambda_1 = 4 - \frac{7 + \sqrt{5}}{2} = \frac{8 - 7 - \sqrt{5}}{2} = \frac{1 - \sqrt{5}}{2}
$$

$$
3 - \lambda_1 = 3 - \frac{7 + \sqrt{5}}{2} = \frac{6 - 7 - \sqrt{5}}{2} = \frac{-1 - \sqrt{5}}{2}
$$

The system:

$$
\begin{bmatrix}
\frac{1 - \sqrt{5}}{2} & 1 \\
1 & \frac{-1 - \sqrt{5}}{2} \\
\end{bmatrix}
\mathbf{v} = 0
$$

From the first row:

$$
\left(\frac{1 - \sqrt{5}}{2} \right) v_1 + v_2 = 0 \Rightarrow v_2 = - \left(\frac{1 - \sqrt{5}}{2} \right) v_1
$$

Choose $v_1 = 1$:

$$
v^{(1)} = \begin{bmatrix}
1 \\
- \frac{1 - \sqrt{5}}{2}
\end{bmatrix}
$$

Normalize $v^{(1)}$:

$$
\| v^{(1)} \| = \sqrt{1^2 + \left(- \frac{1 - \sqrt{5}}{2} \right)^2}
$$

Similarly, find the eigenvector for λ_2, and normalize.

Constructing the Orthogonal Matrix P

Once both eigenvectors are normalized, form P:

$$
P = \begin{bmatrix}
| & | \\
\mathbf{v}_1 & \mathbf{v}_2 \\
| & | \\
\end{bmatrix}
$$

Ensure that P is orthogonal, i.e., columns are orthonormal.

Computing the Diagonal Matrix D

The matrix D is:

$$
D = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
\end{bmatrix}
= \begin{bmatrix}
\frac{7 + \sqrt{5}}{2} & 0 \\
0 & \frac{7 - \sqrt{5}}{2} \\
\end{bmatrix}
$$

Summary of the Diagonalisation Process

- Identify if the matrix A is symmetric.
- Calculate eigenvalues from the characteristic polynomial.
- Find corresponding eigenvectors.
- Normalize eigenvectors to ensure orthogonality.
- Form the orthogonal matrix P with eigenvectors as columns.
- Construct the diagonal matrix D with eigenvalues.
- Verify the diagonalisation: $P^T A P = D$.

Applications of Orthogonal Diagonalisation in Science and Engineering

Understanding how to diagonalise matrices orthogonally has far-reaching applications:

- Principal Component Analysis (PCA) in data science for dimensionality reduction.
- Quantum Mechanics where Hermitian operators are diagonalized to find eigenstates.
- Vibration Analysis in mechanical engineering to determine natural frequencies.
- Control Systems for stability analysis using eigenvalues and eigenvectors.
- Image Processing for filtering and compression techniques.

Conclusion

Frequently Asked Questions

How do you find an orthogonal matrix P that diagonalizes a symmetric matrix A?

To find an orthogonal matrix P that diagonalizes a symmetric matrix A, first compute the eigenvalues and eigenvectors of A. Normalize the eigenvectors to form an orthonormal basis. Arrange these eigenvectors as columns in P. The resulting matrix P is orthogonal, and $P^T AP$ will be a diagonal matrix D containing the eigenvalues.

What is the significance of the matrix D in the diagonalization process?

The matrix D is a diagonal matrix whose entries are the eigenvalues of A. It represents the matrix A in a basis of its eigenvectors, simplifying many matrix operations such as powers and exponentials, and revealing the matrix's spectral properties.

Given matrix A with eigenvalues 10, 4, 2, 2, how do you construct the diagonal matrix D?

Arrange the eigenvalues along the diagonal of D in the same order as their corresponding eigenvectors in P. For the eigenvalues 10, 4, 2, 2, D will be a diagonal matrix: $D = \mathrm{diag}(10, 4, 2, 2)$.

How do you verify that P is an orthogonal matrix in the diagonalization process?

Verify that P is orthogonal by checking if $P^T P$ equals the identity matrix. This confirms that the columns of P are orthonormal eigenvectors, ensuring P is orthogonal.

What are the steps to compute P and D when diagonalizing a given matrix A?

The steps include: 1) Find eigenvalues of A by solving $\det(A - \lambda I) = 0$. 2) For each eigenvalue, find the corresponding eigenvectors. 3) Normalize eigenvectors to form an orthonormal basis. 4) Assemble these eigenvectors as columns to form P. 5) Form D with eigenvalues on the diagonal. 6) Confirm that P is orthogonal and verify $P^T AP = D$.

Why is the matrix P orthogonal when diagonalizing a real symmetric matrix A?

Because symmetric matrices have real eigenvalues and their eigenvectors can be chosen to be orthogonal, the matrix P composed of these eigenvectors is orthogonal, satisfying $P^T P = I$. This orthogonality simplifies the diagonalization process and ensures stability.

Additional Resources

Orthogonal Matrices and Diagonalization: Unlocking the Secrets of Eigenvalues and Eigenvectors

Diagonalization is a fundamental concept in linear algebra, with far-reaching applications across engineering, physics, computer science, and data analysis. Among the various methods to diagonalize matrices, using an orthogonal matrix stands out due to its stability and computational advantages. In this comprehensive review, we explore the process of finding an orthogonal matrix P that diagonalizes a given matrix A, leading to the diagonal matrix D, where $D = P^T A P$.

Understanding the Foundations: Matrices, Eigenvalues, and Eigenvectors

Before diving into the specifics of orthogonal diagonalization, it’s essential to revisit core concepts that underpin the process.

What is a Matrix?

A matrix is a rectangular array of numbers arranged in rows and columns. It serves as a compact way to represent linear transformations, systems of equations, or data structures. For our purposes, matrices like A often represent transformations in a vector space.

Eigenvalues and Eigenvectors

Eigenvalues (λ) and eigenvectors ($\mathbf{v}$) are critical in understanding the behavior of matrices. They satisfy the relation:

$$
A \mathbf{v} = \lambda \mathbf{v}
$$

This equation indicates that applying A to $\mathbf{v}$ results in a scalar multiple of $\mathbf{v}$. The eigenvectors are directions in the vector space that remain invariant under the transformation, scaled by their eigenvalues.

Why are eigenvalues and eigenvectors important?
They allow us to decompose complex transformations into simpler, understandable components. When a matrix is diagonalized, it’s expressed in a basis of its eigenvectors, simplifying many matrix operations.

Orthogonal Matrices: Definition and Significance

What is an Orthogonal Matrix?

An orthogonal matrix P is a square matrix whose transpose equals its inverse:

$$
P^T P = P P^T = I
$$

where I is the identity matrix. This property implies that orthogonal matrices preserve dot products, angles, and lengths—making them ideal for rotations and reflections in geometric transformations.

Properties of Orthogonal Matrices

- Preservation of Lengths and Angles: Orthogonal transformations do not distort the shape of objects—they only rotate or reflect them.
- Determinant: The determinant of an orthogonal matrix is either +1 or -1.
- Eigenvalues: Typically, eigenvalues of orthogonal matrices are complex numbers with magnitude 1, especially in the case of rotations.

Diagonalization of a Matrix: The Core Process

Diagonalization involves finding a matrix P composed of eigenvectors of A, such that:

$$
A = P D P^{-1}
$$

where D is a diagonal matrix containing the eigenvalues of A.

Special case: When A is symmetric, the matrix P can be chosen to be orthogonal ($P^T = P^{-1}$). This orthogonal diagonalization simplifies computations and ensures numerical stability.

Step-by-Step Guide to Finding an Orthogonal Diagonalization

Suppose we are given a symmetric matrix A. Our goal is to find an orthogonal matrix P such that:

\[
D = P^T A P
\]

where \(D \) is diagonal with eigenvalues on the diagonal.

Step 1: Find Eigenvalues of \(A \)

- Compute the characteristic polynomial:

\[
\det(A - \lambda I) = 0
\]

- Solve for \(\lambda \): Find roots of the polynomial, which are the eigenvalues.

Step 2: Find Eigenvectors Corresponding to Each Eigenvalue

- For each eigenvalue \(\lambda_i \), solve:

\[
(A - \lambda_i I) \mathbf{v}_i = 0
\]

- The solutions \(\mathbf{v}_i \) are the eigenvectors.

Step 3: Orthonormalize the Eigenvectors

- Since \(A \) is symmetric, eigenvectors corresponding to different eigenvalues are orthogonal.
- Use the Gram-Schmidt process to orthonormalize the set of eigenvectors, ensuring each vector has unit length.

Step 4: Form the Orthogonal Matrix \(P \)

- Arrange the orthonormal eigenvectors as columns in \(P \):

\[
P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n]
\]

- As the eigenvectors are orthonormal, \(P \) satisfies \(P^T P = I \).

Step 5: Construct the Diagonal Matrix \(D \)

- Place the eigenvalues on the diagonal:

\[
D = \begin{bmatrix}
\lambda_1 & 0 & \dots & 0 \\

0 & \lambda_2 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \lambda_n
\end{bmatrix}
\]

Step 6: Verify the Diagonalization

- Confirm:

\[
A = P D P^T
\]

- Alternatively, check that:

\[
D = P^T A P
\]

Applying the Method: An Example with a Symmetric Matrix

Suppose we have the matrix:

\[
A = \begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\]

Step 1: Find eigenvalues

\[
\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{bmatrix} = (4 - \lambda)^2 - 1 = 0
\]

\[
(4 - \lambda)^2 = 1
\]

\[
4 - \lambda = \pm 1
\]

$$
\Rightarrow \lambda_1 = 4 - 1 = 3, \quad \lambda_2 = 4 + 1 = 5
$$

Step 2: Find eigenvectors

- For $\lambda_1 = 3$:

$$
(A - 3I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

Solve $(A - 3I) \mathbf{v} = 0$:

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0
$$

which yields:

$$
v_1 + v_2 = 0 \Rightarrow v_2 = -v_1
$$

Choose $v_1 = 1$, then $v_2 = -1$. Normalize:

$$
\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
$$

- For $\lambda_2 = 5$:

$$
(A - 5I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
$$

Solve:

$$
- v_1 + v_2 = 0 \Rightarrow v_2 = v_1
$$

Choose $v_1 = 1$, $v_2 = 1$, normalize:

$$
\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Step 3: Form the orthogonal matrix P:

$$

$$
P = \left[\mathbf{v}_1 \quad \mathbf{v}_2 \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
$$

Step 4: Diagonal matrix D:

$$
D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}
$$

Step 5: Verify:

$$
A = P D P^T
$$

which confirms the orthogonal diagonalization.

Implications and Applications of Orthogonal Diagonalization

Orthogonal diagonalization is not just a theoretical exercise; it has practical significance across numerous domains:

- Principal Component Analysis (PCA): In data science, PCA involves diagonalizing covariance matrices to identify principal components.
- Quantum Mechanics: Eigenstates of Hermitian operators (which are symmetric matrices in real space) are orthogonal and form a basis for quantum states.
- Engineering: Vibration analysis often involves diagonalizing stiffness or mass matrices to understand mode shapes.
- Computer Graphics: Rotations and reflections are represented by orthogonal matrices, enabling efficient transformations.

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