Question 2: Solve The Following By Laplace Transforms (a) D? 2 Dt Dax Dx + X = 1 Dt X(0) = X'(0) = 0

Question 2: Solve The Following By Laplace Transforms (a) D? 2 Dt Dax Dx + X = 1 Dt X(0) = X'(0) = 0

Introduction

Solving differential equations is a fundamental aspect of mathematical analysis, especially in engineering, physics, and applied sciences. Among various methods, the Laplace transform stands out as a powerful technique for transforming complex differential equations into algebraic equations that are easier to solve. This article provides a detailed step-by-step solution to the differential equation:

$$ D^2_t X + a D_x X + X = 1 $$

with initial conditions:

$$ X(0) = 0, \quad X'(0) = 0 $$

using Laplace transforms.

Understanding the Problem

Before diving into the solution, it's crucial to understand the components of the problem:

The Differential Equation

$$
D_t^2 X + a D_x X + X = 1
$$

- $D_t^2 X$ represents the second derivative of X with respect to t.
- $D_x X$ is the first derivative of X with respect to x.
- a is a constant coefficient.
- The right side is a constant 1.

Initial Conditions

\[

$$
X(0) = 0, \quad X'(0) = 0
$$

These specify the initial state of the system at $t = 0$.

Step-by-Step Solution Using Laplace Transforms

Step 1: Recognize the Type of Equation

The presence of derivatives with respect to both t and x suggests this is a partial differential equation (PDE). However, since the problem asks to solve by Laplace transforms focusing on t, we treat x as a parameter unless specified otherwise.

If the equation involves derivatives with respect to both variables, the solution approach often involves applying the Laplace transform in t, converting derivatives with respect to t into algebraic terms, simplifying the problem.

Step 2: Apply Laplace Transform in t

Recall:

$$
\mathcal{L}\{X(t)\} = \tilde{X}(s) = \int_0^{\infty} e^{-st} X(t) dt
$$

and the Laplace transform of derivatives:

$$
\mathcal{L}\{X'(t)\} = s \tilde{X}(s) - X(0)
$$

$$
\mathcal{L}\{X''(t)\} = s^2 \tilde{X}(s) - s X(0) - X'(0)
$$

Given initial conditions $X(0) = 0$, $X'(0) = 0$, the transforms simplify to:

$$
\mathcal{L}\{X'(t)\} = s \tilde{X}(s)
$$

$$
\mathcal{L}\{X''(t)\} = s^2 \tilde{X}(s)
$$

Step 3: Transform the PDE

Applying Laplace transform to the entire PDE with respect to t:

$$
\mathcal{L}\left\{ D_t^2 X + a D_x X + X \right\} = \mathcal{L}\{1\}
$$

which becomes:

$$
s^2 \tilde{X}(s) + a D_x \tilde{X}(s) + \tilde{X}(s) = \frac{1}{s}
$$

Note that $D_x \tilde{X}(s)$ is the derivative of $\tilde{X}(s)$ with respect to x, since the transform is in t only.

Step 4: Rearrange the Transformed Equation

Expressed as an ordinary differential equation (ODE) in x:

$$
a \frac{d \tilde{X}(s)}{dx} + (s^2 + 1) \tilde{X}(s) = \frac{1}{s}
$$

This is a first-order linear ODE in x for $\tilde{X}(s, x)$.

Solving the Resulting ODE in x

Step 5: Write the ODE in Standard Form

$$
\frac{d \tilde{X}}{dx} + \frac{s^2 + 1}{a} \tilde{X} = \frac{1}{a s}
$$

Step 6: Find the Integrating Factor

The integrating factor $\mu(x)$:

$$
\mu(x) = e^{\int \frac{s^2 + 1}{a} dx} = e^{\frac{s^2 + 1}{a} x}
$$

Step 7: Integrate and Solve for $\tilde{X}(s, x)$

Multiply through by the integrating factor:

$$
e^{\frac{s^2 + 1}{a} x} \frac{d \tilde{X}}{dx} + e^{\frac{s^2 + 1}{a} x} \frac{s^2 + 1}{a} \tilde{X} = \frac{1}{a s} e^{\frac{s^2 + 1}{a} x}
$$

which simplifies to:

$$
\frac{d}{dx} \left(e^{\frac{s^2 + 1}{a} x} \tilde{X} \right) = \frac{1}{a s} e^{\frac{s^2 + 1}{a} x}
$$

Integrate both sides with respect to x:

$$
e^{\frac{s^2 + 1}{a} x} \tilde{X} = \int \frac{1}{a s} e^{\frac{s^2 + 1}{a} x} dx + C
$$

The integral:

$$
\int e^{\frac{s^2 + 1}{a} x} dx = \frac{a}{s^2 + 1} e^{\frac{s^2 + 1}{a} x} + K
$$

So,

$$
e^{\frac{s^2 + 1}{a} x} \tilde{X} = \frac{1}{a s} \times \frac{a}{s^2 + 1} e^{\frac{s^2 + 1}{a} x} + C
$$

$$
e^{\frac{s^2 + 1}{a} x} \tilde{X} = \frac{1}{s(s^2 + 1)} e^{\frac{s^2 + 1}{a} x} + C
$$

Dividing both sides by the exponential:

$$
\tilde{X}(s, x) = \frac{1}{s(s^2 + 1)} + C e^{-\frac{s^2 + 1}{a} x}
$$

Step 8: Apply Boundary Conditions

Assuming the problem specifies initial conditions at $x = 0$:

$$
X(t, 0) = X_0(t)
$$

If initial conditions are not specified in x, then the constant C is determined by boundary conditions or physical considerations. For simplicity, assume:

$$
\tilde{X}(s, 0) = \tilde{X}_0(s)
$$

From the solution:

$$
\tilde{X}(s, 0) = \frac{1}{s(s^2 + 1)} + C
$$

Suppose the boundary condition at $x=0$ is $X(t, 0) = 0$, then:

$$
\tilde{X}_0(s) = 0
$$

which implies:

$$
0 = \frac{1}{s(s^2 + 1)} + C
$$

Thus,

$$
C = - \frac{1}{s(s^2 + 1)}
$$

Step 9: Write the Final Solution in Laplace Domain

$$
\boxed{
\tilde{X}(s, x) = \frac{1}{s(s^2 + 1)} - \frac{1}{s(s^2 + 1)} e^{-\frac{s^2 + 1}{a} x}
}
$$

Step 10: Inverse Laplace Transform

To obtain $X(t, x)$, perform the inverse Laplace transform:

$$
X(t, x) = \mathcal{L}^{-1} \left\{ \tilde{X}(s, x) \right\}
$$

which involves two terms:

$$
X(t, x) = \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 1)} \right\} -
$$

\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 1)} e^{-\frac{s^2 + 1}{a} x} \right

Frequently Asked Questions

How do you apply Laplace transforms to solve the differential equation $D^2X/Dt^2 + a\, D\, X/Dt + X = 1$ with initial conditions $X(0) = 0$ and $X'(0) = 0$?

First, take the Laplace transform of each term, using the properties for derivatives: $L\{D^2X/Dt^2\} = s^2X(s) - sX(0) - X'(0)$, and $L\{D\, X/Dt\} = sX(s) - X(0)$. Substitute initial conditions, then solve for $X(s)$, and finally find the inverse Laplace transform to get $X(t)$.

What is the Laplace transform of the differential equation $D^2X/Dt^2 + a\, D\, X/Dt + X = 1$ given initial conditions?

Applying Laplace transform, we get $s^2X(s) - sX(0) - X'(0) + a(sX(s) - X(0)) + X(s) = 1$. With $X(0) = 0$ and $X'(0) = 0$, this simplifies to $(s^2 + a s + 1) X(s) = 1$.

How do you find the solution X(t) once you have X(s) in terms of s?

Express $X(s)$ as a rational function, then perform inverse Laplace transform using partial fraction decomposition to obtain $X(t)$. For this equation, $X(s) = 1 / (s^2 + a s + 1)$.

What is the inverse Laplace transform of $1 / (s^2 + a s + 1)$?

The inverse Laplace transform depends on the discriminant of the quadratic denominator. If the quadratic has complex roots, express in terms of exponentials and sine/cosine functions: $X(t) = e^{(-a/2) t} [C_1 \cos(\omega t) + C_2 \sin(\omega t)]$, where $\omega = \sqrt{1 - (a^2/4)}$. Since initial conditions are zero, the particular solution is zero, and the homogeneous solution applies.

What is the general solution for X(t) for the differential equation $D^2X/Dt^2 + a\, D\, X/Dt + X = 1$ with zero initial conditions?

The complete solution is $X(t) = e^{(-a/2) t} [C_1 \cos(\omega t) + C_2 \sin(\omega t)]$ + particular solution. Since the right side is constant, the particular solution is $X_p = 1 / (1)$, which is a constant, but considering initial conditions, the homogeneous solution dominates, leading to the final expression after applying inverse Laplace transform.

Additional Resources

Question 2: Solve the Differential Equation Using Laplace Transforms — An In-Depth Analysis

In the realm of differential equations, the Laplace transform stands out as a powerful technique for solving linear ordinary differential equations (ODEs), especially those with initial conditions and nonhomogeneous terms. The problem at hand — a second-order differential equation involving derivatives with respect to time and a parameter a — exemplifies the practical application of this method. This article provides a comprehensive, detailed examination of how to approach, analyze, and solve the given problem using Laplace transforms, elucidating each step with clarity and insight.

Understanding the Differential Equation

Statement of the Problem

The differential equation we are tasked with solving is:

$$
a \frac{d^2x}{dt^2} + \frac{d x}{dt} + x = 1
$$

with initial conditions:

$$
x(0) = 0, \quad x'(0) = 0
$$

Here, a is a constant parameter, and $x(t)$ is the unknown function of time t.

This is a second-order linear nonhomogeneous differential equation with constant coefficients and an external forcing term (the constant 1). The initial conditions specify the state of the system at $t=0$.

Significance of the Problem and the Methodology

Why Use Laplace Transforms?

Laplace transforms convert differential equations in the time domain into algebraic equations in the complex frequency domain (the s-domain). This transformation simplifies the process of solving differential equations, especially when initial conditions are involved, as it inherently accounts for these conditions within the algebraic framework.

Moreover, the Laplace transform method is particularly advantageous for equations with constant coefficients, discontinuous or impulsive forcing functions, and initial-value problems. It allows for straightforward handling of derivatives, turning them into polynomial expressions in s, thus facilitating an easier solution process.

The Broader Context

This problem encapsulates core concepts in engineering and applied mathematics, such as modeling physical systems (mass-spring-damper systems, electrical circuits) where second-order differential equations frequently arise. Mastery of solving such equations using Laplace transforms is essential for analyzing transient behaviors, stability, and response characteristics of systems.

Step-by-Step Solution Process

Step 1: Write the Differential Equation in Standard Form

The given differential equation is:

$$
a x''(t) + x'(t) + x(t) = 1
$$

where $x''(t)$ denotes the second derivative of $x(t)$ with respect to t, and similarly for $x'(t)$.

The initial conditions are:

$$
x(0) = 0, \quad x'(0) = 0
$$

Step 2: Take the Laplace Transform of Both Sides

Recall the Laplace transform of derivatives:

$$
\mathcal{L}\{x'(t)\} = sX(s) - x(0)
$$
$$
\mathcal{L}\{x''(t)\} = s^2 X(s) - s x(0) - x'(0)
$$

Applying the Laplace transform to each term:

$$
a \left[s^2 X(s) - s x(0) - x'(0) \right] + \left[s X(s) - x(0) \right] + X(s) = \frac{1}{s}
$$

Note that the right side is the Laplace transform of the constant 1, which is $1/s$.

Substituting the initial conditions:

$$
x(0) = 0, \quad x'(0) = 0
$$

we get:

$$
a (s^2 X(s)) + s X(s) + X(s) = \frac{1}{s}
$$

Step 3: Combine Like Terms and Solve for $X(s)$

Factor $X(s)$:

$$
X(s) [a s^2 + s + 1] = \frac{1}{s}
$$

\]

Thus,

\[
X(s) = \frac{1}{s [a s^2 + s + 1]}
\]

This algebraic expression in the \(s \)-domain encapsulates the entire problem.

Analyzing the Algebraic Expression

Understanding the Structure of \(X(s) \)

The expression:

\[
X(s) = \frac{1}{s (a s^2 + s + 1)}
\]

is a rational function with a simple pole at \(s=0 \) and additional poles determined by the roots of the quadratic \(a s^2 + s + 1 = 0 \).

The behavior of \(x(t) \) depends on the nature of these roots, which are influenced by the parameter \(a \).

Factorization and Roots of the Denominator

The quadratic in the denominator:

\[
a s^2 + s + 1 = 0
\]

has roots:

\[
s_{1,2} = \frac{-1 \pm \sqrt{1 - 4 a}}{2 a}
\]

The discriminant:

$$
\Delta = 1 - 4 a
$$

determines whether the roots are real and distinct, real and repeated, or complex conjugates.

- If $a < \frac{1}{4}$, roots are real and distinct.
- If $a = \frac{1}{4}$, roots are real and repeated.
- If $a > \frac{1}{4}$, roots are complex conjugates.

Understanding these roots is crucial for partial fraction decomposition and inverse Laplace transforms.

Partial Fraction Decomposition

Purpose and Procedure

To invert $X(s)$ back into $x(t)$, we perform partial fraction decomposition. The goal is to express $X(s)$ as a sum of simpler fractions whose inverse transforms are well-known.

The general form depends on the nature of roots:

- For distinct real roots, the decomposition involves terms like $\frac{A}{s}$, $\frac{B}{s - s_1}$, $\frac{C}{s - s_2}$.
- For complex roots, the form involves conjugate pairs, leading to exponential and sinusoidal functions upon inversion.

Decomposition Example (Generic Case)

Assuming roots are distinct (for simplicity):

$$
X(s) = \frac{1}{s (a s^2 + s + 1)} = \frac{A}{s} + \frac{B}{s - s_1} + \frac{C}{s - s_2}
$$

Multiply both sides by the denominator:

$$
1 = A (a s^2 + s + 1) + B s (a s^2 + s + 1) / (s - s_1) + C s (a s^2 + s + 1) / (s - s_2)
$$

Alternatively, more straightforward approaches involve expressing $X(s)$ as:

$$
X(s) = \frac{A}{s} + \frac{B s + C}{a s^2 + s + 1}
$$

and then solving for A, B, C.

Inverse Laplace Transform: Deriving $x(t)$

Case 1: Roots are Real and Distinct

If the quadratic factors into real roots s_1, s_2, the inverse transform involves exponential functions:

$$
x(t) = \text{Inverse Laplace of } X(s) \rightarrow \text{a sum of exponentials}
$$

- The term $\frac{A}{s}$ transforms to a step function or constant.
- The terms $\frac{B}{s - s_i}$ transform to $B e^{s_i t}$.

Thus, the solution comprises a constant component plus exponential terms, representing system responses such as overdamped behavior.

Case 2: Roots are Complex Conjugates

If roots are complex conjugates $\alpha \pm i \beta$, the inverse transform yields oscillatory responses:

$$
x(t) = \text{constant} + e^{\alpha t} \left(P \cos \beta t + Q \sin \beta t \right)
$$

This scenario reflects underdamped systems, common in physical oscillators.

Physical Interpretation and Applications

Understanding the solution's structure provides insight into the system's behavior:

- Transient Response: The exponential terms dictate how the system responds initially and whether it settles or oscillates.
- Steady-State Response: The particular solution (steady component) is the constant 1, indicating the system approaches a constant value over time.
- Damping and Oscillations: The nature of roots affects whether the system is overdamped, underdamped, or critically damped

Question 2 Solve The Following By Laplace Transforms A D 2 Dt Dax Dx X 1 Dt X 0 X 0 0

Question 2 Solve The Following By Laplace Transforms A D 2 Dt Dax Dx X 1 Dt X 0 X 0 0

Related Articles

- Refers To The Process Of Translating Instructions Into Signals The Computer Can Execute. A. Decoding
- The 9 Starting Members Of The Baseball Team Are Lining Up For A Picture. What Is The Probability The
- Pls Help! Pls Help! Tap The Image To See The Picture. Do Not Spam Or Else I Will Report!Raghavan Has

Back to Home