Show That The Set Is Linearly Dependent By Finding A Nontrivial Linear Combination Of Vectors In The

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Understanding linear dependence is fundamental in linear algebra, especially when analyzing sets of vectors. Demonstrating that a set of vectors is linearly dependent involves finding a nontrivial linear combination of these vectors that results in the zero vector. This process not only provides insight into the relationships among the vectors but also helps in identifying the basis of vector spaces, understanding their dimension, and solving systems of linear equations. In this article, we will explore the concept of linear dependence in depth, outline methods to identify it, and work through detailed examples to illustrate how to find a nontrivial linear combination that confirms dependence.

Understanding Linear Dependence and Independence

Definition of Linear Dependence

A set of vectors $\{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \}$ in a vector space V over a field $\mathbb{F}$ (such as $\mathbb{R}$ or $\mathbb{C}$) is said to be linearly dependent if there exist scalars $c_1, c_2, \dots, c_k$, not all zero, such that:

$$
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k = \mathbf{0}
$$

In this case, the vectors are dependent because one or more vectors can be expressed as a linear combination of others.

Conversely, if the only solution to this equation is the trivial solution $c_1 = c_2 = \dots = c_k = 0$, then the set is called linearly independent.

Significance of Linear Dependence

- Simplification of vector sets: Identifying linear dependence allows us to eliminate redundant vectors, leading to a minimal spanning set.
- Determining basis and dimension: A basis is a set of linearly independent vectors that span a vector space. Knowing which vectors are dependent helps in constructing bases.
- Solving systems of equations: Linear dependence indicates the presence of infinitely many solutions or relations among vectors.

Methods to Show That a Set Is Linearly Dependent

Establishing linear dependence involves finding a nontrivial linear combination that results in the zero vector. Several methods can be employed:

1. Direct Construction of a Nontrivial Linear Combination

- Step 1: Write the linear combination as an equation:

$$
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k = \mathbf{0}
$$

- Step 2: Solve for the scalars c_i.

- Step 3: If a solution exists where at least one $c_i \neq 0$, the set is linearly dependent.

2. Using Matrix Methods (Row Reduction)

- Step 1: Arrange the vectors as columns in a matrix A.
- Step 2: Perform row operations to reduce A to its row echelon form.
- Step 3: Determine whether the columns are linearly independent:

- If any column can be written as a linear combination of others (i.e., the matrix does not have full rank), the set is dependent.
- The presence of free variables in the solution to $A \mathbf{c} = \mathbf{0}$ indicates dependence.

3. Checking for Zero or Duplicate Vectors

- The presence of the zero vector in the set automatically indicates dependence.
- Duplicate vectors also imply dependence since one vector is a scalar multiple (specifically, the scalar 1) of itself.

Step-by-Step Example: Demonstrating Dependence by Finding a Nontrivial Linear Combination

Let's consider a concrete example to illustrate how to find a nontrivial linear combination that proves the set's dependence.

Example:

Suppose we have the set of vectors in \(\mathbb{R}^3 \):

\[
\mathbf{v}_1 = (1, 2, 3), \quad \mathbf{v}_2 = (2, 4, 6), \quad \mathbf{v}_3 = (0, 1, 1)
\]

Question: Is this set linearly dependent? If yes, find the specific linear combination.

Step 1: Set Up the Linear Combination

We seek scalars \(c_1, c_2, c_3 \) such that:

\[
c_1 (1, 2, 3) + c_2 (2, 4, 6) + c_3 (0, 1, 1) = (0, 0, 0)
\]

which expands component-wise to:

\[
\begin{cases}
c_1 \cdot 1 + c_2 \cdot 2 + c_3 \cdot 0 = 0 \\

c_1 \cdot 2 + c_2 \cdot 4 + c_3 \cdot 1 = 0 \\
c_1 \cdot 3 + c_2 \cdot 6 + c_3 \cdot 1 = 0
\end{cases}
\]

Step 2: Write the System of Equations

\[
\begin{cases}
c_1 + 2 c_2 + 0 c_3 = 0 \quad (1) \\
2 c_1 + 4 c_2 + c_3 = 0 \quad (2) \\
3 c_1 + 6 c_2 + c_3 = 0 \quad (3)
\end{cases}
\]

Step 3: Solve the System

From equation (1):

\[
c_1 = -2 c_2
\]

Substitute into equation (2):

\[
2(-2 c_2) + 4 c_2 + c_3 = 0 \Rightarrow -4 c_2 + 4 c_2 + c_3 = 0 \Rightarrow 0 + c_3 = 0 \Rightarrow c_3 = 0
\]

Now, substitute \(c_1 = -2 c_2 \) and \(c_3 = 0 \) into equation (3):

\[
3(-2 c_2) + 6 c_2 + 0 = 0 \Rightarrow -6 c_2 + 6 c_2 = 0
\]

which simplifies to:

$$
0 = 0
$$

This indicates infinitely many solutions depending on c_2. For example, choose $c_2 = 1$:

$$
c_1 = -2 \times 1 = -2, \quad c_3 = 0
$$

Thus, a nontrivial solution is:

$$
(c_1, c_2, c_3) = (-2, 1, 0)
$$

which satisfies the original linear combination:

$$
-2 \mathbf{v}_1 + 1 \mathbf{v}_2 + 0 \mathbf{v}_3 = \mathbf{0}
$$

Conclusion: The Set Is Linearly Dependent

Since we found a nontrivial linear combination:

$$
-2 \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}
$$

with scalars not all zero ($c_1 = -2, c_2 = 1, c_3 = 0$), the set of vectors is linearly dependent.

Additional Techniques for Verifying Dependence

Using Determinants in 2D and 3D

- For three vectors in $\mathbb{R}^3$, compute the determinant of the matrix formed by them. If the determinant is zero, the vectors are linearly dependent.

Example:

$$
\det \begin{bmatrix}
v_{11} & v_{12} & v_{13} \\
v_{21} & v_{22} & v_{23} \\
v_{31} & v_{32} & v_{33}
\end{bmatrix} = 0 \Rightarrow \text{dependence}
$$

- For two vectors, the cross product can be used to check dependence: if the cross product is zero, the vectors are linearly dependent.

Recognizing Dependence in Infinite Sets

- Infinite sets of vectors often contain dependent vectors, especially if they are scalar multiples or linear combinations of other vectors within the set.

Implications of Showing Linear Dependence

Proving that a set is linearly dependent has several important consequences:

- Redundancy: The vectors in such a set are not all necessary to span the sub

Frequently Asked Questions

How can I determine if a set of vectors is linearly dependent by finding a **nontrivial linear combination?**

To determine if a set is linearly dependent, you need to find scalars, not all zero, such that their linear combination equals the zero vector. If such scalars exist, the set is linearly dependent.

What steps should I follow to show that a set of vectors is linearly dependent using a nontrivial combination?

First, set up the linear combination equation with unknown scalars. Then, solve the resulting system of equations. If you find a solution where not all scalars are zero, the set is linearly dependent.

Can you provide an example of finding a nontrivial linear combination to prove dependence?

Certainly! For vectors v1 = (1, 2), v2 = (2, 4), note that 2v1 = v2, so the combination 2v1 - v2 = 0 shows they are linearly dependent with a nontrivial combination (scalars 2 and -1).

Why does finding a nontrivial linear combination imply that a set of vectors is linearly dependent?

Because it shows that at least one vector in the set can be expressed as a linear combination of the others, indicating the vectors are not all linearly independent.

Is it necessary for the nontrivial linear combination to involve all vectors in the set?

No, it is not necessary. Showing a nontrivial combination involving any subset of vectors suffices to prove the entire set is linearly dependent.

Additional Resources

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Introduction

In the realm of linear algebra, understanding the dependence or independence of a set of vectors is fundamental. This concept underpins many advanced topics, including vector spaces, bases, and matrix theory. When analyzing a set of vectors, mathematicians seek to determine whether there exists a nontrivial linear combination of these vectors that results in the zero vector. If such a combination exists, the set is linearly dependent; if not, it is linearly independent.

This article provides a detailed exploration of how to demonstrate that a set of vectors is linearly dependent by explicitly constructing a nontrivial linear combination. The discussion will include formal definitions, step-by-step procedures, illustrative examples, and analytical insights, making it a comprehensive resource for students and practitioners alike.

Understanding Linear Dependence and Independence

What Is Linear Dependence?

A set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space V over a field $\mathbb{F}$ is linearly dependent if there exist scalars $c_1, c_2, \dots, c_k$, not all zero, such that:

$$
c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0
$$

This means at least one vector in the set can be expressed as a linear combination of others, indicating redundancy within the set.

Conversely, the set is linearly independent if the only scalars satisfying the above equation are all zero, i.e.,

$$
c_1 = c_2 = \dots = c_k = 0
$$

Significance of Showing Dependence

Demonstrating linear dependence is crucial because:

- It reveals that the vectors are not all necessary to span the subspace they occupy.
- It helps identify bases, which require linear independence.
- It simplifies computations involving these vectors, such as solving systems or transforming matrices.
- It informs about the rank of matrices formed from these vectors.

Understanding how to explicitly find a nontrivial linear combination provides concrete proof of dependence and insights into the structure of the vector set.

Formal Framework for Showing Dependence

The Linear Combination Equation

Given vectors $v_1, v_2, \dots, v_k$, the general linear combination is:

$$
c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0
$$

To show that the set is linearly dependent, we need to find specific scalars $\{c_i\}$, not all zero, satisfying this equation.

Approach to Finding the Combination

1. Set up the equation based on the vectors.
2. Express the vectors explicitly, often in coordinate form.
3. Formulate the linear combination equation as a system of linear equations.
4. Solve the system for the scalars $\{c_i\}$.
5. Identify a nontrivial solution, i.e., at least one $c_i \neq 0$.

If such a solution exists, the set is linearly dependent.

Step-by-Step Procedure for Demonstrating Dependence

Step 1: Write Vectors Explicitly

Express each vector v_i in component form, e.g., in $\mathbb{R}^n$:

$$
v_i = (v_{i1}, v_{i2}, \dots, v_{in})
$$

Step 2: Set Up the Linear System

Construct the linear combination:

$$
c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0

\]

which translates to a system:

$$
\begin{cases}
c_1 v_{11} + c_2 v_{21} + \dots + c_k v_{k1} = 0 \\
c_1 v_{12} + c_2 v_{22} + \dots + c_k v_{k2} = 0 \\
\vdots \\
c_1 v_{1n} + c_2 v_{2n} + \dots + c_k v_{kn} = 0
\end{cases}
$$

This is a homogeneous system of linear equations.

Step 3: Solve the System

Use methods such as:

- Gaussian elimination
- Matrix row operations
- Determinant-based analysis (for small systems)

to find solutions for $\{c_i\}$.

Step 4: Identify Nontrivial Solution

- Check if the trivial solution $\{c_i = 0\}$ is the only one.
- If nontrivial solutions (where some $c_i \neq 0$) exist, then the set is linearly dependent.

Step 5: Conclude Dependence

The existence of a nontrivial solution confirms the set's dependence. The specific $\{c_i\}$ form the explicit linear combination demonstrating dependence.

Illustrative Examples

Example 1: Vectors in $\mathbb{R}^3$

Consider the vectors:

$$
v_1 = (1, 2, 3), \quad v_2 = (2, 4, 6), \quad v_3 = (0, 1, 0)
$$

Objective: Show that $\{v_1, v_2, v_3\}$ are linearly dependent.

Step 1: Write the linear combination:

$$
c_1 (1, 2, 3) + c_2 (2, 4, 6) + c_3 (0, 1, 0) = (0, 0, 0)
$$

Step 2: Set up the system:

$$
\begin{cases}
c_1 \cdot 1 + c_2 \cdot 2 + c_3 \cdot 0 = 0 \\
c_1 \cdot 2 + c_2 \cdot 4 + c_3 \cdot 1 = 0 \\
c_1 \cdot 3 + c_2 \cdot 6 + c_3 \cdot 0 = 0
\end{cases}
$$

which simplifies to:

$$
\begin{cases}
c_1 + 2 c_2 = 0 \quad (1) \\
2 c_1 + 4 c_2 + c_3 = 0 \quad (2) \\
3 c_1 + 6 c_2 = 0 \quad (3)
\end{cases}
$$

Step 3: Solve the system:

From (1): $c_1 = -2 c_2$

From (3): $3 c_1 + 6 c_2 = 0 \Rightarrow 3(-2 c_2) + 6 c_2 = 0 \Rightarrow -6 c_2 + 6 c_2 = 0$

which simplifies to $0 = 0$, always true, meaning the third equation doesn't add new constraints.

From (2): $2 c_1 + 4 c_2 + c_3 = 0$

Substitute $c_1 = -2 c_2$:

$$
2(-2 c_2) + 4 c_2 + c_3 = 0 \Rightarrow -4 c_2 + 4 c_2 + c_3 = 0 \Rightarrow 0 + c_3 = 0
$$

Thus, $c_3 = 0$.

Step 4: Express solutions:

$$
c_1 = -2 c_2, \quad c_3 = 0
$$

Choose $c_2 = 1$:

$$
c_1 = -2, \quad c_2 = 1, \quad c_3 = 0
$$

which is a nontrivial solution:

$$
-2 v_1 + 1 v_2 + 0 v_3 = 0
$$

or explicitly,

$$
-2 (1, 2, 3) + (2, 4, 6) + 0 = (0, 0, 0)
$$

This confirms the set is linearly dependent because we've found a nontrivial combination.

Example 2: Vectors in $\mathbb{R}^2$

Suppose:

$$
v_1 = (1, 2), \quad v_2 = (2, 4)
$$

The linear combination:

\[

c_1 (1, 2) + c_2 (2, 4) = (0, 0)
\]

leads to:

\[
\begin{cases}
c_1 + 2 c_2 = 0 \\
2 c_1 + 4 c_2 = 0
\end{cases}
\]

From the first: \(c_1 = -2 c_2\)

Substitute into second: \(2(-2 c_2) +

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