Show That The Two Given Sets Have Equal Cardinality By Describing A Bijection From One To The Other.

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Understanding how to determine whether two sets have the same cardinality is a fundamental concept in set theory and mathematics as a whole. When two sets are said to have equal cardinality, it means there exists a bijection—a one-to-one and onto function—between them. Constructing such a bijection provides a concrete proof that the two sets have the same number of elements, regardless of whether they are finite or infinite. This article offers a comprehensive guide to demonstrating the equality of the cardinalities of two sets through explicit bijections, supported by definitions, examples, and practical techniques.

Understanding Cardinality and Bijections

What Is Cardinality?

Cardinality is a measure of the "size" or number of elements of a set. For finite sets, the cardinality corresponds simply to the number of elements. For example, the set {1, 2, 3} has a cardinality of 3. For infinite sets, the concept extends through the notion of countability and uncountability:

- Countably Infinite Sets: Sets whose elements can be listed in a sequence (like natural numbers, integers, rational numbers). These have the same cardinality as the set of natural numbers, denoted □□ (aleph-null).

- Uncountably Infinite Sets: Sets that cannot be listed in a sequence, such as the set of real numbers between 0 and 1. These sets have a larger cardinality, such as the continuum □.

Determining whether two sets have equal cardinality involves finding an explicit bijection between them, especially in the context of infinite sets.

What Is a Bijection?

A bijection is a function that is both:

- Injective (One-to-One): Each element of the domain maps to a unique element in the codomain; no two elements in the domain map to the same element in the codomain.

- Surjective (Onto): Every element of the codomain has a pre-image in the domain; the function covers the entire codomain.

Mathematically, a function $f: A \to B$ is a bijection if:

- For all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

- For every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

When such a bijection exists, sets A and B are said to have equal cardinality: $|A| = |B|$.

Strategies for Demonstrating Equal Cardinality via Bijections

Constructing a bijection between two sets can sometimes be straightforward, especially when dealing with finite sets, but it can be more challenging in the case of infinite sets. Here are general strategies:

1. Direct Construction of a Bijection

- When possible, explicitly define a function $f: A \to B$ that pairs elements of A with elements of B.

- Verify the function is both injective and surjective.

- This approach provides a concrete proof of equal cardinality.

2. Use of Known Bijections or Isomorphisms

- Leverage existing bijections between familiar sets (e.g., $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$).

- For example, the set of integers $\mathbb{Z}$ is bijective with the natural numbers $\mathbb{N}$, demonstrating they have the same cardinality.

3. Establishing Countability

- Show that a set is countable by constructing a bijection with $\mathbb{N}$.

- For uncountable sets, demonstrate that no such bijection exists (not the focus here; instead, we focus on constructing bijections when possible).

4. Using Set Operations and Decompositions

- Break complicated sets into simpler components for which bijections are known or easier to construct.

- Use union, intersection, or Cartesian product to build bijections from simpler bijections.

Examples of Showing Sets Have Equal Cardinality via Bijections

Example 1: Finite Sets

Suppose $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. To show $|A| = |B|$, define the function:

$$
f: A \to B, \quad f(a) = 1, \quad f(b) = 2, \quad f(c) = 3
$$

This is clearly a bijection because:

- Injective: No two elements of A map to the same element in B.

- Surjective: Every element in B is mapped from some element in A.

Hence, $|A| = |B| = 3$.

Example 2: Countably Infinite Sets

Sets: $\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integers).

Goal: Show $\mathbb{Z}$ has the same cardinality as $\mathbb{N}$.

Bijection Construction:

Define $f: \mathbb{N} \to \mathbb{Z}$ as:

$$
f(n) =
\begin{cases}
\frac{n}{2}, & \text{if } n \text{ is even} \\
-\frac{n-1}{2}, & \text{if } n \text{ is odd}
\end{cases}
$$

Verification:

- For $n = 1$, $f(1) = -0 = 0$.

- For $n = 2$, $f(2) = 1$.

- For $n = 3$, $f(3) = -1$.

- For $n = 4$, $f(4) = 2$.

- And so on.

This function is:

- Injective: Different n produce different integers.

- Surjective: Every integer $z \in \mathbb{Z}$ is obtained for some n.

Thus, $\mathbb{N} \sim \mathbb{Z}$.

Example 3: Rational Numbers and Natural Numbers

The set of rational numbers $\mathbb{Q}$ is countable, despite seeming more complex.

Construction:

- List all rational numbers in a systematic way (e.g., using a diagonal argument).

- Map each rational number to a unique natural number.

This process involves creating a bijection between $\mathbb{Q}$ and $\mathbb{N}$, demonstrating their equal cardinality.

Dealing with Infinite Sets: Common Techniques

Constructing bijections between infinite sets often involves clever enumeration or pairing strategies.

Diagonalization Method

- Used famously by Cantor to prove $\mathbb{R}$ is uncountable.

- For countable sets, a similar enumeration approach can be used to list elements systematically.

Pairing Elements

- For sets like $\mathbb{N} \times \mathbb{N}$, define bijections with $\mathbb{N}$ by pairing elements (e.g., Cantor pairing function).

- Example:

$$
\pi(k, l) = \frac{(k + l)(k + l + 1)}{2} + l
$$

which maps pairs of natural numbers to a single natural number bijectively.

Function Construction via Patterned Mappings

- Establish rules to map elements based on their position or properties.

Key Theorems and Results in Set Cardinality

- Schroeder-Bernstein Theorem: If there exist injective functions $f: A \to B$ and $g: B \to A$, then there exists a bijection $h: A \to B$. This is crucial because sometimes establishing injections is easier than direct bijections.

- Countability and Bijections: Sets are countable if and only if there exists a bijection with $\mathbb{N}$.

- Equivalence of Infinite Cardinalities: Demonstrates that different infinite sets can have the same cardinality via explicit bijections.

Practical Tips for Constructing Bijections

- Start with Known Bijections: Use familiar mappings between common sets.

- Decompose Complex Sets: Break down complicated sets into simpler components with known bijections.

- Use Pairing Functions: For Cartesian products, pairing functions like Cantor's provide an explicit bijection.

- Check Both Conditions: Confirm that the function is injective and surjective through careful proofs.

- Leverage Symmetries: Symmetries and patterns

Frequently Asked Questions

What does it mean for two sets to have equal cardinality?

Two sets have equal cardinality if there exists a bijection between them, meaning a one-to-one and onto function that pairs each element of one set with exactly one element of the other set.

How can we demonstrate that two infinite sets have the same cardinality?

We can demonstrate their equal cardinality by explicitly constructing a bijection—a function that is both injective and surjective—from one set to the other.

What is a bijection and why is it important in comparing set sizes?

A bijection is a function that pairs each element of one set with a unique element of another set, with no elements left unpaired in either set. It is crucial because its existence proves the sets have the same cardinality.

Can you give an example of a bijection between the set of natural numbers and the set of even numbers?

Yes. The function $f(n) = 2n$ maps each natural number n to the even number $2n$, and it is both injective and surjective, establishing a bijection and showing these sets have the same cardinality.

What is the significance of constructing a bijection when comparing

finite sets?

Constructing a bijection directly shows that the finite sets have the same number of elements, confirming their equal cardinality without counting elements individually.

How does defining a bijection help in proving that two sets are equipotent?

Defining a bijection provides a concrete demonstration that the two sets are equipotent, meaning they have the same size, regardless of whether they are finite or infinite.

Additional Resources

Show That The Two Given Sets Have Equal Cardinality By Describing A Bijection From One To The Other

Understanding how to establish the equality of cardinalities between two sets is a foundational concept in set theory, underpinning much of modern mathematics. When two sets are said to have the same cardinality, it implies there exists a perfect pairing between their elements—each element of one set corresponds to exactly one element of the other, and vice versa. This pairing is formalized through the notion of a bijection. In this comprehensive discussion, we explore what it means for two sets to have equal cardinality, how to construct a bijection, and delve into various examples and techniques that illustrate this process.

Understanding Cardinality and Bijections

What Is Cardinality?

Cardinality is a measure of the "size" of a set, indicating how many elements it contains. For finite sets, this is straightforward—the cardinality is simply the number of elements. For example:

- Set $A = \{1, 2, 3\}$ has a cardinality of 3.
- Set $B = \{\text{a, b, c}\}$ also has a cardinality of 3.

When dealing with infinite sets, the concept of cardinality becomes more nuanced. Infinite cardinalities are not just "larger than" finite sizes but are categorized into different sizes, such as countably infinite and uncountably infinite.

Finite sets: Two finite sets A and B are equal in cardinality if $|A| = |B|$.

Infinite sets: Two infinite sets A and B have the same cardinality if there exists a bijection $f: A \to B$.

What Is a Bijection?

A bijection is a function that is both injective (one-to-one) and surjective (onto). Formally:

- Injective (one-to-one): For all $x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Surjective (onto): For all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

A bijection combines these properties, ensuring a perfect pairing between elements of the two sets:

> A bijection from set A to set B is a function $f: A \to B$ such that every element of B is

matched with exactly one element of A, and vice versa.

Implication: If such a function exists, then the sets are said to have equal cardinality. Conversely, if two sets have the same cardinality, there must be a bijection between them.

Constructing a Bijection: A Step-by-Step Approach

Establishing that two sets have equal cardinality hinges on explicitly constructing a bijection. This process can vary widely depending on the nature of the sets involved, but certain general strategies and principles apply.

Step 1: Understand the Sets Involved

Before attempting to construct a bijection, thoroughly analyze the sets:

- Are they finite or infinite?
- Are they subsets of a familiar space (like $\mathbb{N}$, $\mathbb{R}$, etc.)?
- Do they have additional structure (like being sequences, intervals, etc.)?

This understanding guides the choice of an appropriate construction method.

Step 2: Identify Known Bijections or Patterns

Leverage existing bijections or well-known functions:

- For finite sets, simply define a pairing (e.g., $f(1)=a$, $f(2)=b$, etc.).
- For countably infinite sets, use enumeration techniques.
- For uncountable sets, consider continuous functions or known transformations.

Step 3: Define the Function Explicitly

Construct the function $f: A \to B$:

- Ensure f is well-defined for every element in A.
- Describe f explicitly, often piecewise or via formulas.

Step 4: Verify Injectivity and Surjectivity

Prove that:

- Injective: No two different elements in A map to the same element in B.
- Surjective: Every element in B is an image of some element in A.

This often involves algebraic manipulations, inequalities, or logical arguments.

Step 5: Conclude the Sets Have Equal Cardinality

If f is both injective and surjective, then f is a bijection, and the sets A and B have the same cardinality.

Examples of Bijections Demonstrating Equal Cardinality

To deepen understanding, let’s explore classic and illustrative examples where explicit bijections are constructed to prove equal cardinality.

Example 1: Finite Sets

Sets: $A = \{1, 2, 3\}$, $B = \{\text{a, b, c}\}$

Bijection:

Define $f: A \to B$ as:

$$
f(1) = \text{a}, \quad f(2) = \text{b}, \quad f(3) = \text{c}
$$

It’s clearly injective and surjective, establishing that $|A| = |B| = 3$.

Example 2: Countably Infinite Sets — Natural Numbers and Integers

Sets: $A = \mathbb{N} = \{1, 2, 3, \dots \}$, $B = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Goal: Show $\mathbb{N}$ and $\mathbb{Z}$ are countably infinite, i.e., have the same cardinality.

Bijection construction:

Define $f: \mathbb{N} \to \mathbb{Z}$:

$$
f(n) =
\begin{cases}
\frac{n}{2} & \text{if } n \text{ is even} \\
-\frac{n-1}{2} & \text{if } n \text{ is odd}
\end{cases}
$$

Explanation:

- For even $n=2k$, $f(n) = k$, covering all positive integers and zero.
- For odd $n=2k-1$, $f(n) = -k$, covering all negative integers.

Verification:

- Injective: Each n maps to a unique integer.

- Surjective: Every integer $z \in \mathbb{Z}$ has a preimage in $\mathbb{N}$.

Thus, a bijection exists, confirming $|\mathbb{N}| = |\mathbb{Z}|$.

Example 3: Rational Numbers and Natural Numbers

Sets: $A = \mathbb{Q}$, $B = \mathbb{N}$

Goal: Show that $\mathbb{Q}$ (the set of rational numbers) is countably infinite, i.e., $|\mathbb{Q}| = |\mathbb{N}|$.

Method:

- Arrange rationals in a two-dimensional grid: numerator and denominator.
- Use a diagonalization process (similar to Cantor's enumeration) to list all rationals in a sequence.

Bijection:

Construct a sequence $q_1, q_2, q_3, \dots$ listing all rationals without repetition, then define $f: \mathbb{N} \to \mathbb{Q}$ as $f(n) = q_n$.

- This sequence covers all rationals, establishing a bijection.

Addressing Infinite Cardinalities: Uncountable Sets

When sets are uncountably infinite, such as the real numbers $\mathbb{R}$, constructing a bijection often involves more advanced techniques.

Example: The Interval $[0,1]$ and $\mathbb{R}$

Claim: There exists a bijection between the interval $[0,1]$ and $\mathbb{R}$, implying they have the same uncountable cardinality.

Construction:

- Use the tangent function:

$$
f: (0, \pi) \to \mathbb{R}, \quad f(x) = \tan(x - \frac{\pi}{2})
$$

- Restricting or extending this function appropriately, you can create a bijection between $[0,1]$ and $\mathbb{R}$.

Alternative: Use Cantor's diagonal argument to show $\mathbb{R}$ has strictly larger cardinality than $\mathbb{N}$, and hence \([

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