# Show That These Functions Are Contractive On The Indicated Intervals. Determine The Best Values Of X

**Show That These Functions Are Contractive On The Indicated Intervals. Determine The Best Values Of X**

Understanding contraction mappings is a fundamental concept in mathematical analysis, especially in the context of fixed point theory and iterative methods. When dealing with functions defined on specific intervals, establishing whether these functions are contractive is crucial for guaranteeing the existence and uniqueness of fixed points, as well as ensuring convergence of iterative algorithms. This article provides a comprehensive guide on how to demonstrate that certain functions are contractive over given intervals and how to determine the optimal values of $x$ that satisfy these conditions. We will explore the theoretical background, step-by-step procedures, illustrative examples, and practical tips to approach these problems effectively.

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## Introduction to Contractive Functions and Fixed Point Theory

## What is a Contractive Function?

A function $f: X \to X$, where $X$ is a metric space equipped with a distance $d$, is called contractive if there exists a constant $c \in [0, 1)$ such that for all $x, y \in X$:

$$
d(f(x), f(y)) \leq c \cdot d(x, y)
$$

This inequality indicates that $f$ brings points closer together, with the factor $c$ quantifying the maximum rate of contraction. Intuitively, applying a contractive function repeatedly will lead the sequence of iterates to converge to a fixed point.

Key properties of contractive functions:

- They reduce the distance between points by at least a factor $c$.
- They guarantee the existence of a unique fixed point in complete metric spaces (Banach Fixed Point Theorem).
- They are fundamental in iterative algorithms, such as successive approximation methods.

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# Understanding the Context and Importance of Showing Functions are Contractive

Contractive functions are central to many areas of mathematics and applied sciences:

- Fixed Point Theorems: The Banach Fixed Point Theorem states that any contractive function on a complete metric space has exactly one fixed point, and iterative application converges to that fixed point.
- Numerical Methods: Ensuring that iterative algorithms are contractive guarantees convergence to solutions of equations.
- Dynamic Systems: Analyzing stability and long-term behavior often involves demonstrating contraction properties.

Knowing whether a function is contractive within a certain interval allows mathematicians and scientists to:

- Confirm convergence of iterative procedures.
- Establish bounds on the rate of convergence.
- Identify optimal points $x$ where the contraction condition is tight or most effective.

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# Step-by-Step Procedure to Show Functions Are Contractive on Intervals

To demonstrate that a function $f$ is contractive on a specific interval $I$, follow these systematic steps:

## Step 1: Define the Function and the Interval

- Clearly specify the function $f(x)$ under consideration.
- Identify the interval $I = [a, b]$ where the contractivity will be tested.

## Step 2: Determine the Distance Metric

- Typically, the standard Euclidean metric $d(x, y) = |x - y|$ is used for real-valued functions.
- Ensure the metric space is complete over $I$ (e.g., $\mathbb{R}$ with the usual metric).

## Step 3: Compute the Derivative or Lipschitz Constant

- For functions differentiable on $I$, compute the derivative $f'(x)$.
- The function $f$ is contractive if the supremum of $|f'(x)|$ over $I$ is less than 1:

$$
L = \sup_{x \in I} |f'(x)| < 1
$$

- If $f$ is not differentiable, consider using the Lipschitz condition directly or other methods such as mean value inequalities.

## Step 4: Verify the Contraction Condition

- Check if $|f'(x)| \leq c < 1$ for all $x \in I$.
- Alternatively, find the Lipschitz constant $c$ directly:

$$
|f(x) - f(y)| \leq c |x - y| \quad \forall x, y \in I
$$

- This involves analyzing the maximum value of $|f'(x)|$ on $I$.

## Step 5: Confirm the Contractive Nature

- If the above conditions are satisfied, conclude that $f$ is contractive on $I$.
- Otherwise, adjust the interval or function as needed.

## Step 6: Determine the Best Values of $x$

- Identify the points $x \in I$ where the contraction constant $c$ is minimized.
- These points often correspond to the maximum of $|f'(x)|$, which determines the tightest bound for contraction.
- The "best" $x$ values are those where the contraction is strongest, i.e., where $|f'(x)|$ attains its minimum.

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# Illustrative Examples

Let's consider some practical examples to solidify the concepts.

## Example 1: Contractivity of $f(x) = \frac{1}{2}x + 1$ on $[0, 2]$

- Function: $f(x) = \frac{1}{2}x + 1$
- Interval: $[0, 2]$
- Step 1: Derivative $f'(x) = \frac{1}{2}$
- Step 2: Since $|f'(x)| = 0.5 < 1$, the function is Lipschitz with constant $c = 0.5$.
- Step 3: Therefore, $f$ is contractive on $[0, 2]$ with contraction factor $c = 0.5$.

Conclusion: The function is contractive on the interval with the best (smallest) $c = 0.5$.

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## Example 2: Contractivity of $g(x) = \cos(x)$ on $[0, \pi/2]$

- Function: $g(x) = \cos(x)$
- Interval: $[0, \pi/2]$
- Step 1: Derivative $g'(x) = -\sin(x)$
- Step 2: The absolute value $|g'(x)| = |\sin(x)|$

- Since $\sin(x)$ is increasing on $[0, \pi/2]$, its maximum is at $x = \pi/2$:

$$
|g'(x)| \leq |\sin(\pi/2)| = 1
$$

- To confirm contraction, need $c < 1$:

$$
\sup_{x \in [0, \pi/2]} |\sin(x)| = 1
$$

- But the supremum equals 1, so $g$ is not contractive on the entire interval because the Lipschitz constant equals 1.

However:

- On a subinterval, say $[0, \pi/3]$, $|\sin(x)| \leq \sin(\pi/3) = \sqrt{3}/2 \approx 0.866 < 1$, so $g$ is contractive with $c \approx 0.866$.

Conclusion: To guarantee contractivity, choose the interval where $|\sin(x)|$ is strictly less than 1, and find the maximum of $|\sin(x)|$ in that interval.

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## Practical Tips for Determining the Best Values of $x$

- Use calculus tools to analyze derivatives and find extrema of $|f'(x)|$ within the interval.
- When direct differentiation is complex, consider numerical methods or graphing to estimate maximum $|f'(x)|$.
- Recognize that the best (tightest) contraction constant occurs at the point where $|f'(x)|$ attains its maximum.
- If the maximum is less than 1, the function is contractive on the entire interval; if it equals 1, contractivity is only guaranteed on a subinterval where the derivative's magnitude is less than 1.

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## Implications and Applications of Showing Functions are Contractive

The process of demonstrating that a function is contractive is not purely theoretical; it has widespread applications:

- Solving Nonlinear Equations: Contractive mappings underpin iterative techniques such as fixed-point iteration and Newton-Raphson methods.
- Computer Algorithms: Ensuring convergence of algorithms in numerical analysis, machine learning, and optimization.
- Stability Analysis: In control systems and dynamical systems, contraction properties imply system stability.
- Mathematical Modeling: Establishing the robustness of models where the iterative process converges to a steady state.

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## Conclusion

Proving that a function is contractive over a specified interval is a fundamental step in many mathematical and applied contexts. By carefully analyzing derivatives, calculating Lipsch

## Frequently Asked Questions

### How can I verify if a function is contractive on a given

## interval?

To verify if a function is contractive, you need to show that there exists a constant k in (0,1) such that for all x and y in the interval, $|f(x) - f(y)| \leq k|x - y|$. This often involves computing the Lipschitz constant or using the Mean Value Theorem to estimate the maximum derivative magnitude on the interval.

## What is the significance of the Lipschitz constant in determining if a function is contractive?

The Lipschitz constant provides an upper bound on how much the function can stretch distances. If the Lipschitz constant is less than 1 on an interval, the function is contractive there, ensuring convergence properties useful in fixed point theorems.

## How do I find the best value of X to make a function contractive on an interval?

Determine the maximum value of the absolute value of the derivative of the function on the interval. The interval's endpoints or critical points are checked, and the largest of these maximum derivatives (less than 1) indicates the interval where the function is contractive. Adjust the interval accordingly to include only points where the derivative's magnitude is below 1.

## Can you provide a step-by-step method to show a function is contractive on a specific interval?

Yes. First, find the derivative of the function. Next, determine its maximum absolute value on the interval. If this maximum (L) is less than 1, then the function is contractive with Lipschitz constant L. Finally, verify the inequality $|f(x) - f(y)| \leq L|x - y|$ holds for all x, y in the interval.

## Are there common functions that are inherently contractive on certain intervals?

Yes. Functions like linear contractions with slopes less than 1, certain exponential decay functions, and some trigonometric functions restricted to specific intervals are inherently contractive when their derivatives' magnitudes are less than 1 on those intervals.

## How does the choice of interval affect whether a function is contractive?

The interval determines where you evaluate the derivative or Lipschitz constant. A function may be contractive on a smaller interval but not on the entire domain. Carefully selecting an interval where the derivative's magnitude is less than 1 ensures the function's contractiveness there.

## What is the role of the Mean Value Theorem in establishing contractiveness?

The Mean Value Theorem relates the difference in function values to the derivative at some point in the interval. It helps in estimating $|f(x) - f(y)| \leq M|x - y|$, where M is the maximum of $|f'(x)|$ on the interval. If $M < 1$, it shows the function is contractive.

## How can I determine the best interval for a function to be contractive?

Analyze the derivative of the function and identify the largest subinterval where $|f'(x)| \leq k < 1$. The interval where this holds is the best candidate for contractiveness. Adjust the interval boundaries to maximize the interval size while maintaining the $|f'(x)| < 1$ condition.

## Why is establishing contractiveness important in solving fixed point problems?

Contractive functions guarantee the existence and uniqueness of fixed points by Banach's Fixed Point Theorem. Showing a function is contractive on an interval allows iterative methods to converge to the fixed point efficiently.

# Additional Resources

Contractive Functions: A Comprehensive Exploration of Their Behavior and Optimal Values of X

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Introduction

In the realm of mathematical analysis, particularly within the study of fixed point theorems and iterative methods, contractive functions hold a place of significant importance. These functions, characterized by their property of bringing points closer together, underpin many algorithms in numerical analysis, optimization, and computer science. Understanding how to verify whether a function is contractive over a particular interval and determining the optimal values of the variable \( x \) where the function exhibits this property is essential for both theoretical insights and practical applications.

This article aims to provide an in-depth, detailed exploration of how to demonstrate that specific functions are contractive on given intervals and how to identify the best (or optimal) values of \( x \) for which this contractivity holds. By adopting an expert tone akin to a product review or advanced tutorial, we will dissect the necessary criteria, employ mathematical tools such as derivatives and Lipschitz constants, and illustrate these concepts through comprehensive examples.

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Understanding Contractive Functions

What Is a Contractive Function?

A function $f: D \to D$, where $D \subseteq \mathbb{R}$, is called contractive (or a contraction mapping) if there exists a constant $0 \leq k < 1$ such that for all $x, y \in D$:

$$
|f(x) - f(y)| \leq k |x - y|
$$

This inequality signifies that the function $f$ does not increase the distance between any two points—it "contracts" the space locally or globally within the domain $D$. The constant $k$ is known as the Lipschitz constant, and its value being less than 1 is the hallmark of contractivity.

Significance in Fixed Point Theorems

The Banach Fixed Point Theorem states that any contraction mapping $f$ on a complete metric space $D$ admits a unique fixed point $x^$, satisfying $f(x^) = x^$. Moreover, iterative methods starting from any initial point $x_0$ in $D$ will converge exponentially fast to $x^$.

Thus, verifying contractivity is not just a theoretical exercise; it ensures the convergence and stability of iterative algorithms used in solving equations numerically.

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Demonstrating Contractivity on an Interval

Step 1: Identify the Domain $D$

The first step involves clearly defining the interval $D$ over which you are testing the function's contractivity. This could be a finite interval such as $[a, b]$ or an open interval $(a, b)$. The choice of domain often stems from the context of the problem or the specific application.

Step 2: Compute the Derivative $f'(x)$

For functions that are differentiable on the interval $D$, the derivative provides critical insight into the function's behavior. Specifically, the Lipschitz constant $k$ can often be bounded by:

$$
k \geq \sup_{x \in D} |f'(x)|
$$

If this supremum is less than 1, then $f$ is a contraction on $D$.

Example: For a differentiable function $f$, if:

\[

$$
\max_{x \in D} |f'(x)| \leq k < 1,
$$

then $f$ is contractive on $D$.

Step 3: Verify the Lipschitz Condition

The core of the proof involves showing that:

$$
|f(x) - f(y)| \leq k |x - y| \quad \forall x, y \in D
$$

which can be established via the Mean Value Theorem. If $f$ is differentiable on $D$, then for any $x, y \in D$, there exists some $c$ between $x$ and $y$ such that:

$$
f(x) - f(y) = f'(c)(x - y)
$$

Taking absolute values:

$$
|f(x) - f(y)| = |f'(c)| |x - y| \leq \sup_{x \in D} |f'(x)| \times |x - y|
$$

Hence, the Lipschitz constant $k$ can be taken as $\sup_{x \in D} |f'(x)|$.

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Determining the Best Values of $x$

The phrase "best values of $x$" in this context refers to those points within the domain $D$ where $f$ exhibits the strongest contractive behavior—that is, where the Lipschitz constant is minimized or the derivative's magnitude is at its lowest.

Why is this important? Because identifying these points helps in:

- Optimizing convergence rates of iterative methods.
- Ensuring contractivity in the widest possible domain.
- Designing functions or algorithms that are robust and efficient.

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Practical Approach to Find These Values

1. Analyze $f'(x)$:

- Find $f'(x)$ over $D$.
- Determine where $|f'(x)|$ attains its minimum in $D$.

This is because the contraction constant $k$ is bounded above by the maximum of $|f'(x)|$. To enhance contractivity, you want the Lipschitz constant to be as small as possible, ideally less than 1.

2. Locate the Minimum of $|f'(x)|$:

- Use calculus techniques: set the derivative of $|f'(x)|$ to zero and analyze critical points.
- Check endpoints if the interval is closed.

3. Confirm the Contractive Condition:

- Verify that at the points where $|f'(x)|$ is minimized, the value is less than 1.
- If the minimal $|f'(x)|$ over $D$ is less than 1, then $f$ is contractive on $D$, possibly with the optimal Lipschitz constant equal to this minimum.

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Extensive Examples

Let's concretize these concepts with detailed examples.

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Example 1: Analyzing $f(x) = \frac{1}{2}x + \frac{1}{4}$

Step 1: Domain Selection

Suppose we are interested in the interval $D = [0, 1]$.

Step 2: Derivative Computation

$$
f'(x) = \frac{1}{2}
$$

which is constant over $D$.

Step 3: Contractivity Verification

Since $|f'(x)| = \frac{1}{2} < 1$, by the Mean Value Theorem:

$$
|f(x) - f(y)| \leq \frac{1}{2} |x - y| \quad \forall x, y \in [0, 1]
$$

Thus, $f$ is a contraction on $[0, 1]$ with $k = \frac{1}{2}$.

Optimal $x$ Values

Because the derivative is constant, the best (smallest) Lipschitz constant is $0.5$, attained everywhere on $D$. Therefore, the function is uniformly contractive, and the best values of $x$ are all points in $D$.

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Example 2: Analyzing $f(x) = \cos(x)$

Step 1: Domain Selection

Suppose $D = [0, \pi/2]$.

Step 2: Derivative Computation

$$
f'(x) = -\sin(x)
$$

Thus,

$$
|f'(x)| = |\sin(x)| = \sin(x)
$$

since $\sin(x) \geq 0$ on $[0, \pi/2]$.

Step 3: Find the Maximum of $|f'(x)|$

- The maximum of $\sin(x)$ on $[0, \pi/2]$ is at $x = \pi/2$:

$$
|f'(\pi/2)| = 1
$$

- The minimum at $x = 0$:

$$
|f'(0)| = 0
$$

- The supremum of $|f'(x)|$ over $D$ is 1.

Implication: Since the Lipschitz constant $k$ is at most 1, $f$ is not a contraction on $[0, \pi/2]$ because $k$ must be strictly less than 1.

However, if we restrict the domain further, say $D' = [0, \pi/3]$, then:

$$
|f'(x)| = \sin(x) \leq \sin(\pi/3) = \frac{\sqrt{3}}{2} \approx 0.866 < 1
$$

Therefore, $f$ is contractive on $D'$ with $k \leq 0.866$.

Best \( x \) values for contractivity are those where \( |f'(x)| \)

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