Solve The Given Differential Equation By Using An Appropriate Substitution. The De Is Homogeneous. Y

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When faced with a differential equation, especially a homogeneous differential equation involving the variable y, selecting the right substitution method can significantly simplify the solving process. In this article, we will explore how to solve a given homogeneous differential equation by using an appropriate substitution, focusing on the substitution $y = vx$ or $x = vy$. These substitutions help transform the original differential equation into a separable or more manageable form, making it easier to find the general solution.

Understanding Homogeneous Differential Equations

What is a Homogeneous Differential Equation?

A differential equation is called homogeneous if it can be expressed in the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

or equivalently, when the right-hand side can be written as a function of the ratio y/x. Homogeneous equations are characterized by their scale invariance, meaning that replacing y with ky and x with kx preserves the form of the equation.

Key features of homogeneous differential equations:
- The right-hand side depends only on the ratio y/x.
- They often appear in physics and engineering problems where proportional relationships are involved.
- They can be transformed into separable equations using substitution techniques.

Why Use Substitution for Homogeneous Equations?

Using substitution simplifies the process of solving homogeneous differential equations because it reduces the original equation to a separable form. Typically, the substitution involves expressing y as a multiple of x (or vice versa), which simplifies the dependence on the variables.

Common substitutions include:
- $y = vx$, where v is a function of x.
- $x = vy$, when more appropriate.

In this article, we focus on the substitution $y = vx$, which is most common for homogeneous equations involving y/x.

Step-by-Step Solution Using the Substitution $y = vx$

1. Recognize the Homogeneous Equation

Identify if the differential equation is homogeneous by checking whether the right side can be expressed as a function of y/x.

For example:

$$\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$$

Rewrite the right side:

$$\frac{2xy + y^2}{x^2} = \frac{2xy}{x^2} + \frac{y^2}{x^2} = 2 \frac{y}{x} + \left(\frac{y}{x} \right)^2$$

Since the right side depends only on y/x, the equation is homogeneous.

2. Make the Substitution $y = vx$

Set:

$$y = vx$$

where v is a function of x. Then, differentiate y:

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

This substitution transforms the original differential equation into an equation in terms of v and x.

3. Rewrite the Differential Equation

Replace y and dy/dx in the original equation:

$$v + x \frac{dv}{dx} = F(v)$$

This often results in a separable differential equation in v and x.

Using the example:

$$ v + x \frac{dv}{dx} = 2v + v^2 $$

Rearranged as:

$$ x \frac{dv}{dx} = 2v + v^2 - v = v + v^2 $$

which simplifies to:

$$ x \frac{dv}{dx} = v (1 + v) $$

4. Separate Variables and Integrate

Express the equation in a separable form:

$$ \frac{dv}{v(1+v)} = \frac{dx}{x} $$

Now, perform partial fraction decomposition on the left:

$$ \frac{1}{v(1+v)} = \frac{A}{v} + \frac{B}{1+v} $$

Solve for A and B:

$$ 1 = A(1 + v) + B v $$

Set $v = 0$:

$$ 1 = A(1) \Rightarrow A = 1 $$

Set $v = -1$:

$$ 1 = B (-1) \Rightarrow B = -1 $$

Thus:

$$ \frac{1}{v(1+v)} = \frac{1}{v} - \frac{1}{1+v} $$

Integrate both sides:

$$ \int \left(\frac{1}{v} - \frac{1}{1+v} \right) dv = \int \frac{1}{x} dx $$

which yields:

$$ \ln |v| - \ln |1+v| = \ln |x| + C $$

or, combining logs:

$$ \ln \left| \frac{v}{1+v} \right| = \ln |x| + C $$

Exponentiating both sides:

$$ \left| \frac{v}{1+v} \right| = K x $$

where $K = e^{C}$.

Expressing the Solution in Terms of y and x

Recall that $v = y / x$. Substitute back:

$$ \left| \frac{\frac{y}{x}}{1 + \frac{y}{x}} \right| = K x $$

Simplify numerator and denominator:

$$ \left| \frac{y / x}{(x + y) / x} \right| = K x $$

$$ \left| \frac{y}{x + y} \right| = K x $$

This implicit solution relates y and x. To find an explicit form, proceed as follows:

$$ \frac{y}{x + y} = \pm K x $$

Cross-multiplied:

$$ y = \pm K x (x + y) $$

$$ y = \pm K x^2 + \pm K x y $$

Bring all y terms to one side:

$$ y - \pm K x y = \pm K x^2 $$

Factor y:

$$ y (1 - \pm K x) = \pm K x^2 $$

Finally, solve for y:

$$ y = \frac{\pm K x^2}{1 - \pm K x} $$

This expression provides the general solution to the original differential equation.

Additional Tips for Solving Homogeneous Differential Equations

1. Always Verify Homogeneity

Before applying substitution, confirm that the differential equation is homogeneous by checking if the right side can be written as a function of y/x.

2. Choose the Correct Substitution

- Use $y = vx$ if the equation involves y/x.
- Use $x = vy$ if the equation involves x/y.

3. Simplify and Integrate Carefully

Partial fractions and substitution are key to simplifying the integral. Always perform these steps carefully to avoid errors.

4. Express the Final Solution Clearly

Depending on the context, you might want to solve explicitly for y or leave the solution in an implicit form.

Conclusion

Solving homogeneous differential equations by using an appropriate substitution, such as $y = vx$, is a powerful technique that simplifies complex equations into manageable forms. Recognizing the structure of the differential equation is crucial, as it guides the choice of substitution. Once the substitution is made, the problem becomes one of separation of variables and integration, leading to the general solution. Mastering this method not only enhances your problem-solving toolkit but also deepens your understanding of the fundamental properties of differential equations, especially those exhibiting scale invariance.

Whether you're working through academic problems or applying these techniques in engineering and physics, understanding how to solve homogeneous differential equations effectively is essential. With practice, identifying the right substitution and executing the steps systematically will become second nature, enabling you to tackle a wide range of differential equations with confidence.

Frequently Asked Questions

What is the first step to solve a homogeneous differential equation using substitution?

The first step is to verify that the differential equation is homogeneous, typically by checking if the functions involved are homogeneous functions of the same degree, then use the substitution y = vx to simplify the equation.

How do you identify if a differential equation is homogeneous?

A differential equation is homogeneous if it can be expressed such that all terms are of the same degree or if it can be written in the form dy/dx = F(y/x) where F is a function of y/x.

What substitution is used to solve a homogeneous differential equation?

The substitution y = vx (or y = ux) is used, where v = y/x to convert the differential equation into a separable form.

After substituting y = vx, what is the next step in solving the differential equation?

Differentiate y = vx to find dy/dx in terms of v and x, then substitute back into the original equation to obtain a separable differential equation in v and x.

How do you solve the resulting equation after substitution in a homogeneous differential equation?

Once in separable form, integrate both sides with respect to the appropriate variables to find v in terms of x, then back-substitute y = vx to find the solution.

Can you give an example of a homogeneous differential equation and how to solve it?

Yes. For example, dy/dx = (x + y) / (x - y). Recognize it as homogeneous, substitute y = vx, leading to an equation in v and x. Simplify and separate variables, then integrate to find the general solution.

What are common mistakes to avoid when solving homogeneous differential equations?

Common mistakes include not verifying the equation is homogeneous, incorrect application of the substitution y = vx, forgetting to differentiate y = vx properly, or failing to separate variables correctly after substitution.

How does the homogeneity of a differential equation simplify the solving process?

Homogeneity allows the substitution y = vx, which reduces the original equation to a separable form, making it easier to integrate directly.

Are all differential equations that look homogeneous solvable using substitution?

Most are, provided they meet the homogeneity criteria. However, some may require additional substitutions or methods if they are not directly separable after the initial substitution.

What are the key properties of a differential equation that make substitution y = vx appropriate?

The key property is that the equation's right-hand side can be expressed as a function of y/x alone, indicating the equation is homogeneous and suitable for substitution y = vx.

Additional Resources

Solve The Given Differential Equation By Using An Appropriate Substitution. The De Is Homogeneous. Y

When approaching differential equations, particularly those that are homogeneous, employing an appropriate substitution can significantly simplify the solving process. Homogeneous differential equations possess a specific structure that allows for strategic substitution, often transforming a complex problem into a more manageable one. This article explores the methodology for solving such equations, emphasizing the importance of recognizing their homogeneous nature and selecting the right substitution to facilitate an efficient solution.

Understanding Homogeneous Differential Equations

Definition and Characteristics

A differential equation is said to be homogeneous if it can be expressed in a form where all terms are of the same degree when considering the variables involved. More formally, a first-order differential equation:

\[\frac{dy}{dx} = F(x, y) \]

is homogeneous if \(F(tx, ty) = F(x, y) \) for all non-zero \(t \), indicating that \(F \) is a homogeneous function of degree zero.

Key features of homogeneous differential equations include:

- The ratio \(\frac{dy}{dx} \) can be expressed as a function of \(y/x \), i.e., \(

$\frac{dy}{dx} = G\left(\frac{y}{x}\right)$.
- They often present in forms where substitution $v = y/x$ simplifies the problem.

Why Recognize Homogeneity?

Identifying the homogeneous nature allows for a targeted substitution strategy, typically $v = y/x$, which reduces the differential equation's complexity. Recognizing this pattern early enables a more straightforward solution process and often avoids unnecessary algebraic manipulations.

Appropriate Substitution for Homogeneous Equations

The Substitution $v = \frac{y}{x}$

The most common and effective substitution for solving homogeneous differential equations is:

$$
v = \frac{y}{x}
$$

which implies

$$
y = v x
$$

Differentiating both sides with respect to x:

$$
\frac{dy}{dx} = v + x \frac{dv}{dx}
$$

This substitution transforms the original differential equation into an equation in terms of v and x, often leading to a separable form.

Advantages of Using $v = y/x$

- Simplifies the differential equation by reducing the number of variables.
- Converts the problem into a separable differential equation, which is easier to solve.
- Universally applicable to all first-order homogeneous equations.

Limitations and Considerations

- The substitution is valid only for homogeneous equations; applying it blindly can lead to complications.
- Sometimes, after substitution, the resulting equation may be more complicated, requiring further manipulation or substitution.

Step-by-Step Solution Process

1. Recognize the Homogeneous Equation

Start by verifying if the differential equation can be written in the form where $F(tx, ty) = F(x, y)$. Check if the right-hand side can be expressed as a function of y/x.

2. Make the Substitution $v = y/x$

Replace y with $v x$, and compute dy/dx accordingly:

$$
dy/dx = v + x dv/dx
$$

3. Rewrite the Differential Equation

Substitute into the original equation to obtain an equation involving v and x:

$$
v + x \frac{dv}{dx} = G(v)
$$

where $G(v)$ is the function of y/x.

4. Separate Variables

Rearranged into a form:

$$
x \frac{dv}{dx} = G(v) - v
$$

which can be written as:

\[

\frac{dv}{G(v) - v} = \frac{dx}{x}
\]

Now, the variables are separable.

5. Integrate Both Sides

Perform the integration:

\[
\int \frac{dv}{G(v) - v} = \int \frac{dx}{x}
\]

which yields:

\[
\int \frac{dv}{G(v) - v} = \ln |x| + C
\]

The integral on the left depends on the specific form of \(G(v) \).

6. Back-Substitute to Find \(y \)

After solving for \(v \), recall \(y = v x \), so:

\[
y = v x
\]

Substitute back the solution for \(v \) to express \(y \) explicitly or implicitly.

Example: Solving a Homogeneous Differential Equation

Suppose we are asked to solve the differential equation:

\[
\frac{dy}{dx} = \frac{y^2 + x^2}{2xy}
\]

Step 1: Recognize Homogeneity

Rewrite numerator and denominator:

\[
\frac{dy}{dx} = \frac{y^2 + x^2}{2xy}
\]

Note that numerator and denominator are homogeneous of degree 2 and 2, respectively, and the entire expression simplifies to a function of \(y/x \).

Step 2: Substitute \(v = y/x \)

\[
y = v x
\]

Differentiate:

\[
dy/dx = v + x dv/dx
\]

Step 3: Rewrite the equation

Express the right side in terms of \(v \):

\[
\frac{dy}{dx} = \frac{(v x)^2 + x^2}{2 v x \cdot x} = \frac{v^2 x^2 + x^2}{2 v x^2}
= \frac{x^2 (v^2 + 1)}{2 v x^2} = \frac{v^2 + 1}{2 v}
\]

Now, equate:

\[
v + x dv/dx = \frac{v^2 + 1}{2 v}
\]

Step 4: Rearrange

\[
x dv/dx = \frac{v^2 + 1}{2 v} - v
\]

Simplify the right side:

\[
\frac{v^2 + 1}{2 v} - v = \frac{v^2 + 1 - 2 v^2}{2 v} = \frac{-v^2 + 1}{2 v}
\]

Thus,

\[
x dv/dx = \frac{1 - v^2}{2 v}
\]

Step 5: Separate variables

$$
\frac{2 v}{1 - v^2} dv = \frac{1}{x} dx
$$

Note that:

$$
\frac{2 v}{1 - v^2} = - \frac{2 v}{v^2 - 1}
$$

But it's easier to handle as is:

$$
\int \frac{2 v}{1 - v^2} dv = \int \frac{1}{x} dx
$$

Step 6: Integrate

Express the left integral:

$$
\int \frac{2 v}{1 - v^2} dv
$$

Use substitution:

$$
u = 1 - v^2 \Rightarrow du = -2 v dv
$$

Therefore:

$$
- \int \frac{1}{u} du = \int \frac{1}{x} dx
$$

Integrate:

$$
- \ln |u| = \ln |x| + C
$$

Back-substitute:

$$
- \ln |1 - v^2| = \ln |x| + C
$$

Rewrite:

$$
\ln |1 - v^2| = - \ln |x| + C'
$$

or

$$
|1 - v^2| = \frac{K}{x}
$$

where $K = e^{C'}$ is an arbitrary constant.

Step 7: Back to y

Recall $v = y/x$, so:

$$
|1 - (y/x)^2| = \frac{K}{x}
$$

which simplifies to:

$$
|x^2 - y^2| = K x
$$

This is an implicit solution to the original differential equation.

Features and Benefits of the Substitution Method

Pros:

- Simplifies complex equations: The substitution reduces the order and complexity, often converting an equation into a separable form.
- Universally applicable for homogeneous equations: The y/x substitution is a standard tool, making the approach systematic.
- Facilitates integration: Once separated, integration is straightforward, especially if standard integral forms are recognized.

Cons:

- Requires recognition: The key step is identifying the homogeneous structure; misclassification can lead to incorrect substitution.
- May lead to complicated integrals: Sometimes, the resulting integrals are not straightforward and require additional techniques.
- Implicit solutions: Frequently, the solution obtained is implicit, requiring further algebraic manipulation or interpretation.
