Suppose Random Variable X Has Probability Density Function F(x)=xe Xfor X>0 And F(x)=0 Otherwise.

Suppose Random Variable X Has Probability Density Function F(x)=xe Xfor X>0 And F(x)=0 Otherwise. This intriguing probability density function (pdf) describes a continuous random variable X that takes positive real values. Understanding the properties of such a distribution involves exploring its fundamental characteristics, deriving important statistics, and examining applications where it might arise. In this comprehensive article, we delve into the details of this distribution, analyze its properties, and provide insights into how it fits within the broader context of probability theory.

Understanding the Given Probability Density Function

Definition of the PDF

The pdf provided is:

$$
f(x) = x e^{x}, \quad \text{for } x > 0
$$

and

$$
f(x) = 0, \quad \text{for } x \leq 0.
$$

At first glance, this function appears to be a product of x and the exponential function e^{x}. However, for a function to be a valid probability density function, it must satisfy two key properties:

1. Non-negativity: $f(x) \geq 0$ for all x.
2. Normalization: $\int_{-\infty}^{\infty} f(x) dx = 1$.

Given $f(x) = x e^{x}$ for $x > 0$, it is positive in this domain, but whether it integrates to 1 needs verification.

Checking Validity of the PDF

To verify if $f(x)$ is a valid pdf, compute the integral:

$$
\int_{0}^{\infty} x e^{x} dx
$$

If this integral equals 1, then the function is a valid density; otherwise, it needs to be normalized.

Let's evaluate this integral:

$$
I = \int_{0}^{\infty} x e^{x} dx
$$

Note that e^{x} grows exponentially, and since x is also increasing, the integral diverges, indicating that the integral is infinite. To see this explicitly, we can attempt integration by parts:

Set $u = x \Rightarrow du = dx$

Set $dv = e^{x} dx \Rightarrow v = e^{x}$

Then,

$$
I = uv - \int v du = x e^{x} - \int e^{x} dx = x e^{x} - e^{x} + C
$$

Evaluated from 0 to ∞:

$$
\lim_{b \to \infty} [b e^{b} - e^{b}] - [0 \cdots e^{0} - e^{0}] = \lim_{b \to \infty} (b e^{b} - e^{b}) - (0 - 1)
$$

As $b \to \infty$, both $b e^{b}$ and e^{b} tend to infinity, with $b e^{b}$ dominating. Therefore, the integral diverges to infinity.

Conclusion: The integral of $f(x) = x e^{x}$ over $(0, \infty)$ is infinite, so this function cannot be a probability density function as is. It is likely that the intended pdf was different, perhaps a typo or misstatement. A common distribution involving x and e^{x} is the gamma distribution or an exponential distribution, but with different parameters.

Note: For the purpose of this discussion, let's assume the correct pdf is:

$$
f(x) = x e^{-x}, \quad x > 0
$$

which is the pdf of a Gamma distribution with shape parameter $k=2$ and rate parameter $\lambda=1$. Alternatively, if the original was meant to be $f(x) = x e^{-x}$, the properties are well known.

In the following, we proceed with:

$$
f(x) = x e^{-x}, \quad x > 0
$$

since this is a valid pdf with known applications.

Assuming the corrected pdf: $f(x) = x e^{-x}$, for $x > 0$

Properties of the Distribution with PDF $f(x) = x e^{-x}$

Validation of the PDF

Let's verify normalization:

$$
\int_{0}^{\infty} x e^{-x} dx
$$

This integral is well-known:

$$
\int_{0}^{\infty} x e^{-x} dx = 1! = 1
$$

since:

$$
\int_{0}^{\infty} x^{n} e^{-\lambda x} dx = \frac{n!}{\lambda^{n+1}}
$$

for positive integers n, with $\lambda=1$. Here, $n=1$, so:

$$
\int_{0}^{\infty} x e^{-x} dx = 1!
= 1
$$

which confirms that $f(x)$ is a valid probability density function.

Connection to Gamma Distribution

The pdf:

$$
f(x) = \frac{\lambda^{k}}{\Gamma(k)} x^{k-1} e^{-\lambda x}

$$
$$

is the gamma distribution with shape k and rate λ. For our case:

$$
f(x) = x e^{-x} = \frac{1^{2}}{\Gamma(2)} x^{2-1} e^{-1 \cdot x} = \frac{1}{\Gamma(2)} x^{1} e^{-x}
$$

Since $\Gamma(2) = 1!\ = 1$, the distribution is a Gamma distribution with parameters:

$$
k = 2, \quad \lambda = 1
$$

Key Statistical Properties

1. Expected Value (Mean)

For a Gamma distribution with parameters k and λ, the mean is:

$$
E[X] = \frac{k}{\lambda}
$$

For our case:

$$
E[X] = \frac{2}{1} = 2
$$

2. Variance

Variance of a Gamma distribution:

$$
Var[X] = \frac{k}{\lambda^{2}}
$$

which gives:

$$
Var[X] = \frac{2}{1^{2}} = 2
$$

3. Moment Generating Function (MGF)

The MGF of a Gamma distribution:

$$
M_{X}(t) = \left(\frac{\lambda}{\lambda - t} \right)^{k}, \quad t < \lambda
$$

For our parameters:

$$
M_{X}(t) = \left(\frac{1}{1 - t} \right)^{2}
$$

valid for $t < 1$.

4. Probability Density Function Summary

Property	Value
Distribution	Gamma ($k=2, \lambda=1$)
Support	$x > 0$
Mean	2
Variance	2
Mode	$k - 1 = 1$ (since $k > 1$)

Applications and Implications

Real-World Applications of the Gamma Distribution

The gamma distribution appears in various fields such as:

- Reliability Engineering: Modeling the lifetime of products with failure rate depending on the number of prior failures.
- Queuing Theory: Describing waiting times and service durations.
- Bayesian Statistics: As a conjugate prior for exponential family distributions.
- Hydrology: Modeling rainfall amounts and other natural phenomena.

Significance of the Distribution's Parameters

The shape parameter $k=2$ indicates the distribution is skewed right, with the mode at $x=1$. The mean at 2 suggests that, on average, the modeled event or quantity tends to be around 2 units.

Deriving Additional Statistics and Functions

1. Cumulative Distribution Function (CDF)

The CDF:

$$
F(x) = P(X \leq x) = \int_{0}^{x} t e^{-t} dt
$$

This integral evaluates to:

$$
F(x) = 1 - (x + 1) e^{-x}
$$

for $x > 0$. This result comes from integration by parts or recognizing the incomplete gamma function.

2. Quantile Function

The inverse of the CDF allows us to find quantiles. For example, the median $x_{0.5}$:

$$
F(x_{0.5}) = 0.5 \Rightarrow 1 - (x_{0.5} + 1) e^{-x_{0.5}} = 0.5
$$

which simplifies to:

$$
(x_{0.5} + 1) e^{-x_{0.5}} = 0.5
$$

Solving for $x_{0.5}$ requires

Frequently Asked Questions

What is the probability density function (pdf) of the random variable X?

The pdf of X is defined as $F(x) = x e^{-x}$ for $x > 0$, and $F(x) = 0$ otherwise.

Is the given function a valid probability density function?

Yes, because the function is non-negative for $x > 0$ and its integral over $(0, \infty)$ equals 1, satisfying the properties of a valid pdf.

How do we find the cumulative distribution function (CDF) of X?

The CDF, denoted as F_X(x), is obtained by integrating the pdf from 0 to x: F_X(x) = $\int_0^x$ t e^{-t} dt.

What is the expected value E[X] of the random variable X?

E[X] can be calculated as E[X] = $\int_0$^∞ x (x e^{-x}) dx = $\int_0$^∞ x^2 e^{-x} dx, which equals 2 by applying the gamma function.

What is the variance Var(X) of this distribution?

Using the moments, Var(X) = E[X^2] - (E[X])^2. Since E[X] = 2 and E[X^2] = 6, the variance is 6 - 2^2 = 2.

Additional Resources

Suppose Random Variable X Has Probability Density Function F(x)=xe^x for X>0 And F(x)=0 Otherwise: An In-Depth Investigation

---

Introduction

In the realm of probability theory and statistical analysis, the characterization of random variables through their probability density functions (PDFs) forms a cornerstone for understanding diverse stochastic phenomena. The function under consideration—Suppose Random Variable X Has Probability Density Function F(x) = x e^x for X > 0 and F(x) = 0 otherwise—presents an intriguing case that warrants detailed exploration. At first glance, the form appears unconventional, raising key questions about its validity, properties, and implications in probabilistic modeling.

This article embarks on a comprehensive analysis of this density function, dissecting its mathematical structure, verifying its legitimacy as a PDF, examining its statistical properties, and exploring potential applications and interpretations within various domains.

---

1. Clarification of the Probability Density Function

The initial statement suggests that the PDF of the random variable \(X\) is

\[
f(x) = x e^x, \quad \text{for } x > 0,
\]
and
\[

f(x) = 0, \quad \text{for } x \leq 0.
\]

However, this form warrants scrutiny. The notation "F(x)" typically denotes the cumulative distribution function (CDF), while the probability density function (PDF) is commonly represented as \(f(x)\). Given the context, it appears the author intends \(f(x) = x e^x\).

Key question: Is \(f(x) = x e^x\) a valid PDF on \((0, \infty)\)?

---

2. Validity as a Probability Density Function

A function qualifies as a PDF if it satisfies two conditions:

- Non-negativity: \(f(x) \geq 0\) for all \(x\) in its domain.
- Normalization: \(\int_{-\infty}^\infty f(x) \, dx = 1\).

Given \(f(x) = x e^x\) for \(x > 0\), and zero elsewhere, let's verify whether these conditions hold.

2.1 Non-negativity

For \(x > 0\):

\[
f(x) = x e^x,
\]

which is clearly positive since \(x > 0\) and \(e^x > 0\). For \(x \leq 0\):

\[
f(x) = 0,
\]

so the non-negativity condition is satisfied across the entire real line.

2.2 Normalization

The integral over the domain must equal 1:

\[
\int_0^{\infty} x e^x \, dx = 1.
\]

Calculate the integral:

\[
I = \int_0^{\infty} x e^x \, dx.
\]

This integral diverges due to the exponential growth, but let's verify explicitly.

---

3. Integral Evaluation and Normalization

3.1 Computing $\int_0^{\infty} x e^x \, dx$

Using integration by parts:

Let:

- $u = x \Rightarrow du = dx$,
- $dv = e^x dx \Rightarrow v = e^x$.

Then:

$$
I = uv - \int v du = x e^x - \int e^x dx = x e^x - e^x + C.
$$

Evaluate from 0 to ∞:

$$
I = \lim_{b \to \infty} [b e^b - e^b] - [0 \cdot e^0 - e^0] = \lim_{b \to \infty} (b e^b - e^b) - (-1).
$$

As $b \to \infty$:

- e^b grows exponentially,
- $b e^b$ also diverges to infinity.

Therefore,

$$
b e^b - e^b \to \infty,
$$

and the integral diverges to infinity. This indicates that

$$
\int_0^{\infty} x e^x dx = \infty,
$$

which cannot be normalized to 1.

---

4. Implication: The Function Is Not a Valid PDF

Since the integral of $f(x) = x e^x$ over $(0, \infty)$ diverges, it cannot serve as a probability density function in its current form. This raises the need for clarification or correction.

---

5. Possible Interpretations and Corrections

Given the divergence, one might consider:

- A typo or misinterpretation: Perhaps the intended density was $f(x) = x e^{-x}$, which is familiar as the PDF of a Gamma distribution with shape parameter 2 and scale 1.
- A different domain: Maybe the domain is bounded or the function is different.
- Alternative functions: Maybe the expression is $f(x) = x e^{-\lambda x}$ with $\lambda > 0$, ensuring integrability.

Let's explore these options.

---

6. The Gamma Distribution Connection

The function $f(x) = x e^{-\lambda x}$, for $x > 0$, is the PDF of a Gamma distribution with shape $\alpha = 2$ and scale $\theta = 1/\lambda$:

$$
f(x) = \frac{x^{\alpha - 1} e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)}.
$$

For $\alpha = 2$, $\theta = 1$:

$$
f(x) = \frac{x^{1} e^{-x}}{\Gamma(2)} = \frac{x e^{-x}}{1!} = x e^{-x}.
$$

This function is integrable over $(0, \infty)$, with integral equal to 1, making it a valid PDF.

Conclusion: It is highly likely that the original function intended was $f(x) = x e^{-x}$, not $x e^{x}$.

---

7. Re-examining the Original Statement

Assumption: The correct density might be

$$
f(x) = x e^{-x} \quad \text{for } x > 0,
$$
and zero otherwise.

This correction aligns with standard probability distributions and ensures the integral over $(0, \infty)$ converges to 1:

$$

$$
\int_0^{\infty} x e^{-x} dx = 1,
$$
since

$$
\int_0^{\infty} x e^{-x} dx = \Gamma(2) = 1!.
$$

---

8. Statistical Properties of the Corrected Distribution

Assuming the corrected density $f(x) = x e^{-x}$:

8.1 Distribution Type

This is the Gamma distribution with shape parameter $\alpha=2$ and scale parameter $\theta=1$.

8.2 Moments

- Expected value:

$$
E[X] = \alpha \theta = 2 \times 1 = 2.
$$

- Variance:

$$
Var(X) = \alpha \theta^2 = 2 \times 1^2 = 2.
$$

- Higher moments:

$$
E[X^n] = \theta^n \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}.
$$

For example:

$$
E[X^2] = 1^2 \frac{\Gamma(4)}{\Gamma(2)} = \frac{3!}{1!} = 6,
$$
which aligns with the variance calculation since:

$$
Var(X) = E[X^2] - (E[X])^2 = 6 - 4 = 2.
$$

8.3 Probability Density Function Summary

Parameter	Value
Distribution	Gamma
Shape (α)	2
Scale (θ)	1
Mean	2
Variance	2

---

9. Deeper Mathematical Analysis

9.1 PDF Behavior

The PDF $f(x) = x e^{-x}$ exhibits:

- Zero at $x=0$,
- Peaks at $x = 1$,
- Decays exponentially as $x \to \infty$.

The mode can be found by setting the derivative to zero:

$$
f'(x) = e^{-x} - x e^{-x} = e^{-x} (1 - x) = 0,
$$
which yields:

$$
x = 1,
$$
confirming the mode is at $x=1$.

9.2 CDF

The cumulative distribution function (CDF):

$$
F(x) = \int_0^x t e^{-t} dt,
$$

which evaluates to:

$$
F(x) = 1 - e^{-x} (x + 1),
$$
for $x \geq 0$.

---

10. Applications and Interpretations

The gamma distribution with shape 2 and scale 1 has applications in:

-

Suppose Random Variable X Has Probability Density Function F X Xe Xfor X Gt 0 And F X 0 Otherwise

Suppose Random Variable X Has Probability Density Function F X Xe Xfor X Gt 0 And F X 0 Otherwise

Related Articles

- Select The Correct Text In The Passage. Which Statements Indicate Inductive Reasoning?1. Tanya Is An
- Q-box Question There Are Many Influences On Agricultural Development In Pakistan .read The Following
- The '5 Why' Method: Select One: A. Is A Brainstorming Tool. B. Seeks To Understand The Root Cause Of

Back to Home