Suppose X Is Uniform Over (-1,1) And Y=X2. Are X AndY Uncorrelated? Are X And Y Independent? Explain

Suppose X Is Uniform Over (-1,1) And Y=X2. Are X AndY Uncorrelated? Are X And Y Independent? Explain

Understanding the relationship between random variables is a fundamental aspect of probability theory and statistics. In this context, we are examining the specific case where X is uniformly distributed over the interval $(-1, 1)$, and $Y = X^2$. The key questions are: are X and Y uncorrelated? Are they independent? To answer these, we need to delve into the concepts of correlation and independence, analyze the properties of these variables, and understand how they relate to each other.

Understanding the Basics: Uncorrelated vs. Independent Variables

Before analyzing the specific case, it is essential to clarify what it means for two random variables to be uncorrelated and what independence entails.

Uncorrelated Random Variables

- Two random variables X and Y are uncorrelated if their covariance is zero:

$$
\operatorname{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = 0
$$

- Covariance measures the linear relationship between the variables. If it is zero, it indicates no linear association, but it does not necessarily imply that the variables are independent.

Independent Random Variables

- Two variables X and Y are independent if the joint probability distribution factors into the product of their marginal distributions:

$$
f_{X,Y}(x, y) = f_X(x) \times f_Y(y)
$$

- Independence implies uncorrelatedness, but the converse is not necessarily true. Variables can be uncorrelated but still dependent through nonlinear relationships.

Analyzing the Distribution of X and Y

The variables in question have specific distributions:

Distribution of X

- X is uniformly distributed over $(-1, 1)$:
$$
f_X(x) = \frac{1}{2} \quad \text{for} \quad -1 < x < 1
$$
- The mean of X is:
$$
E[X] = 0
$$
- The variance of X is:
$$
Var(X) = \frac{(b - a)^2}{12} = \frac{(1 - (-1))^2}{12} = \frac{4}{12} = \frac{1}{3}
$$

Distribution of $Y = X^2$

- Since $Y = X^2$, and X is uniform over $(-1, 1)$, Y takes values in $[0, 1)$.
- The distribution of Y can be derived from the distribution of X:
- For $y \in [0,1)$, the values of X satisfying $X^2 = y$ are $X = \pm \sqrt{y}$.
- The probability density function (pdf) of Y is:
$$
f_Y(y) = f_X(\sqrt{y}) \left| \frac{d}{dy} \sqrt{y} \right| + f_X(-\sqrt{y}) \left| \frac{d}{dy} (-\sqrt{y}) \right| = 2 \times \frac{1}{2} \times \frac{1}{2 \sqrt{y}} = \frac{1}{2 \sqrt{y}}
$$
for $0 < y < 1$.

Are X and Y Uncorrelated?

To determine if X and Y are uncorrelated, we need to compute their covariance:

$$
\operatorname{Cov}(X, Y) = E[XY] - E[X]E[Y]
$$

Since $E[X] = 0$, the covariance simplifies to:

$$
\operatorname{Cov}(X, Y) = E[XY]

$$

Calculating $E[XY]$

- Recall $Y = X^2$, so:

$$
E[XY] = E[X \times X^2] = E[X^3]
$$

- For a uniform distribution over $(-1, 1)$:

$$
E[X^3] = \int_{-1}^{1} x^3 \times \frac{1}{2} dx
$$

- Because x^3 is an odd function and the interval is symmetric:

$$
E[X^3] = 0
$$

Therefore,

$$
\operatorname{Cov}(X, Y) = 0
$$

This indicates that X and Y are uncorrelated.

Are X and Y Independent?

While uncorrelatedness is established, independence is a stronger condition that requires the joint distribution to factor into marginals.

Examining the Joint Distribution $f_{X,Y}(x, y)$

- Since $Y = X^2$, the joint distribution is concentrated along the parabola $y = x^2$.
- For a fixed x, Y takes the value x^2 with probability density:

$$
f_{X,Y}(x, y) = f_X(x) \delta(y - x^2)
$$

where δ is the Dirac delta function.

Key point: The joint distribution does not factor into $f_X(x) \times f_Y(y)$ because Y is fully determined by X. The distribution of Y depends on the value of X, making the two variables dependent.

Implication of Dependence

- Since Y is a deterministic function of X, the variables are not independent.
- The dependence is nonlinear: knowing X directly determines Y, but knowing Y alone does not determine X uniquely, as both $x = \pm \sqrt{y}$.

Conclusion: X and Y are uncorrelated but dependent.

Summary of Findings

To encapsulate the analysis:

- Since X is uniformly distributed over $(-1, 1)$, $E[X] = 0$ and $Var(X) = 1/3$.

- Given $Y = X^2$, the covariance between X and Y is zero because $E[X^3] = 0$. Therefore, they are uncorrelated.

- However, the relationship $Y = X^2$ is deterministic, meaning the joint distribution does not factor into the product of marginals, which implies that X and Y are dependent.

- The dependence is nonlinear; knowing X gives full knowledge of Y, but knowing Y does not specify X uniquely.

Implications in Statistical Analysis

Understanding whether variables are uncorrelated or independent impacts statistical modeling and inference:

- Uncorrelated but dependent variables: This scenario shows that lack of linear correlation does not imply independence. Nonlinear relationships can exist, and models assuming independence may be invalid.
- Practical considerations: In data analysis, observing zero covariance should not be taken

as evidence of independence. Additional tests or model specifications are necessary to establish independence.

Conclusion

In summary, when X is uniformly distributed over $(-1, 1)$ and $Y = X^2$, the variables are uncorrelated since their covariance is zero. However, they are not independent because Y is completely determined by X, establishing a dependency. This illustrates an important principle in probability theory: uncorrelated variables are not necessarily independent. Recognizing the distinction is crucial in statistical modeling, data analysis, and understanding the underlying relationships between variables.

Additional Notes and Extensions

- Nonlinear dependence: The dependence between X and Y is nonlinear, which can be detected through nonlinear measures of association beyond covariance.
- Generalization: Similar reasoning applies to other functions of random variables, where deterministic functions induce dependence even if the variables are uncorrelated.
- Applications: This concept is relevant in fields like econometrics, machine learning, and signal processing, where understanding the nature of relationships influences model choice and interpretation.

In conclusion, analyzing the relationship between X and $Y = X^2$ demonstrates the importance of distinguishing between uncorrelatedness and independence in probability and statistics. While

Frequently Asked Questions

If X is uniformly distributed over (-1,1) and Y = X^2, are X and Y uncorrelated?

Yes, X and Y are uncorrelated because the covariance between X and Y is zero due to the symmetry of the uniform distribution around zero, which makes the cross-term vanish.

Are X and Y independent when Y = X^2 and X ~

Uniform(-1,1)?

No, X and Y are not independent because Y is a deterministic function of X; knowing X completely determines Y.

Why are X and Y uncorrelated but not independent in this scenario?

They are uncorrelated because the covariance is zero, but they are not independent because Y is directly determined by X, indicating a functional dependence.

How can we verify whether X and Y are uncorrelated in this case?

By calculating the covariance Cov(X, Y) and showing that it equals zero, which confirms they are uncorrelated.

Does the symmetry of X's distribution influence the correlation between X and Y?

Yes, the symmetry of the uniform distribution about zero causes the expected value of XY to be zero, leading to zero covariance and hence uncorrelation.

Is the fact that $Y = X^2$ affect the independence between X and Y?

Yes, since Y is a function of X, knowing X completely determines Y, which means they cannot be independent.

Can two variables be uncorrelated but still dependent? Illustrate with X and Y here.

Yes, in this case, X and Y are uncorrelated because their covariance is zero, but they are dependent since Y is a deterministic function of X.

What is the significance of X and Y being uncorrelated but not independent in statistical analysis?

It indicates that while there is no linear relationship between them, there may still be a functional or non-linear dependence, which is important in understanding their relationship.

Additional Resources

Suppose X Is Uniform Over (-1,1) And $Y=X^2$. Are X And Y Uncorrelated? Are X And Y Independent? Explain

Understanding the relationship between random variables is a foundational aspect of probability theory and statistics. When analyzing variables such as X, which is uniformly distributed over the interval (-1, 1), and Y, defined as the square of X (i.e., $Y = X^2$), it's essential to explore whether these variables are uncorrelated or independent. This exploration reveals important insights into their joint behavior, dependence structure, and the implications for statistical modeling and inference. In this article, we delve into the concepts of correlation and independence, analyze the specific case of X and Y, and clarify the distinctions with thorough explanations and illustrative calculations.

Understanding the Problem Setup

Let's begin by summarizing the key elements:

- X is a continuous random variable with a uniform distribution over the interval (-1, 1). Its probability density function (pdf) is:

$$
f_X(x) = \frac{1}{2} \quad \text{for} \quad x \in (-1, 1)
$$

and zero elsewhere.

- Y is defined as $Y = X^2$. Since X takes values in (-1, 1), Y takes values in (0, 1).

The central question is:

- Are X and Y uncorrelated?
- Are X and Y independent?

To answer these, we'll need to look at their joint distribution, marginal distributions, and compute the correlation coefficient where appropriate.

Analyzing the Relationship Between X and Y

Before directly addressing uncorrelatedness and independence, it's instructive to understand the nature of the variables:

- Y is a deterministic function of X. Specifically, Y is completely determined once X is known because $Y = X^2$.
- The relationship is symmetric around zero; both $X = \pm \sqrt{Y}$ satisfy the equation.

This deterministic relationship has a significant implication: it often indicates potential dependence, but not necessarily correlation. We will examine this precisely.

Are X and Y Uncorrelated?

Understanding Uncorrelatedness

Two random variables X and Y are said to be uncorrelated if their covariance is zero:

$$
\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0
$$

Note that uncorrelatedness does not imply independence unless the variables are jointly Gaussian. In general, variables can be uncorrelated but still dependent.

Calculating Expectations

Let's compute the necessary expectations:

1. E[X]

Given the uniform distribution over (-1, 1):

$$
E[X] = \int_{-1}^{1} x \cdot f_X(x) \, dx = \int_{-1}^{1} x \cdot \frac{1}{2} \, dx = \frac{1}{2} \left[\frac{x^2}{2} \right]_{-1}^{1} = \frac{1}{2} \left(\frac{1^2}{2} - \frac{(-1)^2}{2} \right) = 0
$$

since $1^2 = (-1)^2 = 1$, so the difference is zero.

2. E[Y]

Since $Y = X^2$:

$$
E[Y] = E[X^2] = \int_{-1}^{1} x^2 \cdot f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x^2 \, dx
$$

Calculate:

$$

$$
E[X^2] = \frac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{2} \left(\frac{1^3}{3} - \frac{(-1)^3}{3} \right) = \frac{1}{2} \left(\frac{1}{3} - \frac{-1}{3} \right) = \frac{1}{2} \left(\frac{2}{3} \right) = \frac{1}{3}
$$

3. E[XY]

Given that $Y = X^2$, the expectation $E[XY] = E[X \cdot X^2] = E[X^3]$.

Calculate:

$$
E[X^3] = \int_{-1}^{1} x^3 \cdot f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x^3 \, dx
$$

Since x^3 is an odd function symmetric about zero:

$$
E[X^3] = \frac{1}{2} \left[\frac{x^4}{4} \right]_{-1}^{1} = \frac{1}{2} \left(\frac{1^4}{4} - \frac{(-1)^4}{4} \right) = \frac{1}{2} (\frac{1}{4} - \frac{1}{4}) = 0
$$

4. Covariance $\text{Cov}(X, Y)$

Putting all together:

$$
\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0 - (0) \times \frac{1}{3} = 0
$$

Result: The covariance between X and Y is zero.

Conclusion on Uncorrelatedness

Since $\text{Cov}(X, Y) = 0$, X and Y are uncorrelated.

However, this does not imply independence. Zero covariance indicates no linear relationship, but the variables could still be dependent in a nonlinear way, which is precisely the case here, as we will see.

Are X and Y Independent?

Understanding Independence

Two variables X and Y are independent if their joint probability distribution factors into the product of their marginals:

$$
f_{X,Y}(x, y) = f_X(x) \times f_Y(y)
$$

for all (x, y) in the support.

In our case, since $Y = X^2$, Y is completely determined once X is known, indicating a deterministic relationship. This strongly suggests dependence, but let's analyze more rigorously.

Deriving the Joint Distribution $f_{X,Y}(x, y)$

Given the transformation $Y = X^2$:

- For $x \in (-1, 1)$, $y = x^2$.

- The joint distribution is concentrated on the curve $y = x^2$ in the (x, y) plane.

Since X is continuous over $(-1, 1)$, the joint distribution is degenerate along the curve $y = x^2$. More precisely, for a fixed $y \in (0, 1)$, the two possible x values are:

$$
x = \pm \sqrt{y}
$$

The joint density can be expressed as:

$$
f_{X,Y}(x, y) = f_X(x) \times \delta(y - x^2)
$$

where δ is the Dirac delta function, indicating that the joint distribution is supported only on the set where $y = x^2$.

Implication: Because the joint distribution is supported only on a nonlinear manifold (the parabola $y = x^2$), X and Y are not independent. The fact that Y is a deterministic function of X ensures dependence.

Formal Explanation of Dependence

- If X and Y were independent, then the joint distribution would factor into marginals. But in

this case, knowing X determines Y exactly, and vice versa (up to the sign of \(\pm \sqrt{Y} \)).
- The structure of the joint distribution confirms dependence; the distribution is degenerate along the curve \(y = x^2 \), which cannot be factored into independent marginals.

Summary of Dependence

- X and Y are not independent because Y is a deterministic function of X.
- The joint distribution is confined to the parabola \(y = x^2 \).
- The dependence is nonlinear, yet the covariance is zero due to symmetry considerations.

Key Takeaways and Implications

- Uncorrelatedness does not imply independence: In this case, X and Y are uncorrelated but dependent. The zero covariance is due to the symmetry of the distribution of X around zero and the odd-power nature of \(X^3 \).
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