The Autocorrelation Function Of A Random Process X(t) Is Given By RXX()=3+9e What Is The Mean Of The

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Introduction to Random Processes and Autocorrelation Function

Understanding Random Processes

A random process, also known as a stochastic process, is a collection of random variables indexed by time or space. It models systems or phenomena where uncertainty or inherent variability exists, such as stock prices, temperature variations, or signal transmissions. Mathematically, a random process $X(t)$ assigns a probability distribution to the set of possible outcomes at each time t.

The Significance of Autocorrelation Function

The autocorrelation function (ACF), $R_{XX}(\tau)$, of a random process $X(t)$ measures the correlation between values of the process at different times separated by lag τ. It provides insights into the temporal dependencies within the process, revealing how the process's current state relates to its past or future states. Formally, for a stationary process,

$$
R_{XX}(\tau) = E[X(t)X(t + \tau)]
$$

where $E[\cdot]$ denotes the expected value.

Given Autocorrelation Function: $R_{XX}(\tau) = 3 + 9e^{-|\tau|}$

Analyzing the Autocorrelation Function

The provided autocorrelation function is:

$$
R_{XX}(\tau) = 3 + 9e^{-|\tau|}

\]

This function combines a constant term and an exponential decay component. Let's interpret its components:

- The constant term, 3, indicates the baseline level of autocorrelation.
- The exponential term, $9e^{-|\tau|}$, describes how the autocorrelation diminishes with increasing $|\tau|$, characteristic of a process with a certain degree of memory that fades over time.

Properties of the Given $RXX(\tau)$

The function exhibits the following properties:

- Symmetry: $RXX(\tau) = RXX(-\tau)$, due to the absolute value.
- Maximum at $\tau=0$: $RXX(0) = 3 + 9e^{0} = 3 + 9 = 12$.
- Decays exponentially as $|\tau|$ increases, approaching the baseline of 3.

Determining the Mean of the Random Process X(t)

Relationship Between Autocorrelation and Mean

For a wide-sense stationary (WSS) process, the autocorrelation function relates to the mean and variance as follows:

$$
R_{XX}(0) = E[X(t)^2] = \text{Var}(X) + (E[X])^2
$$

The autocorrelation at zero lag, $RXX(0)$, provides the second moment of the process.

The mean, $\mu = E[X(t)]$, is related to the autocorrelation function through the following considerations:

- If the process has a non-zero mean, the autocorrelation function can be expressed in terms of the mean and the autocovariance.
- For a stationary process with mean μ, the autocorrelation function can be written as:

$$
R_{XX}(\tau) = \text{Cov}(X(t), X(t + \tau)) + \mu^2
$$

- When $\tau=0$,

$$
R_{XX}(0) = \text{Var}(X) + \mu^2
$$

Note: If the process is stationary and the autocorrelation function is known, the mean can often be extracted if the autocorrelation at zero lag is known, and the variance is determined or assumed.

Applying to Our Given RXX(τ)

Given:

$$
R_{XX}(0) = 3 + 9e^{-0} = 3 + 9 = 12
$$

Thus,

$$
R_{XX}(0) = E[X(t)^2] = 12
$$

From the properties, if the process has a non-zero mean μ, then:

$$
E[X(t)^2] = \text{Var}(X) + \mu^2
$$

However, without explicit information about the variance, we need to analyze the autocorrelation function further.

Identifying the Mean of the Process

Implications of the Autocorrelation Function Structure

The autocorrelation function's form suggests that the process may include a deterministic component or a constant mean.

- The constant term (3) can be interpreted as the autocorrelation of the process's fluctuations about its mean.
- The exponential decay term ($9e^{-|\tau|}$) indicates the presence of a correlation structure typical of a first-order autoregressive process.

Using the Limit of RXX(τ) as τ → ∞

As |τ| approaches infinity:

$$
R_{XX}(\tau) \rightarrow 3
$$

This limiting value is often associated with the autocovariance of the fluctuations about the mean, i.e., the variance of the zero-mean component of the process.

Assuming the process is stationary with mean μ, then:

$$
R_{XX}(\tau) = \text{Cov}(X(t), X(t + \tau)) + \mu^2
$$

At large |τ|, the covariance term tends to zero:

$$
\lim_{|\tau|\to\infty} R_{XX}(\tau) = \mu^2
$$

Given that,

$$
\lim_{|\tau|\to\infty} R_{XX}(\tau) = 3
$$

Therefore,

$$
\mu^2 = 3
$$

$$
\Rightarrow \mu = \pm \sqrt{3}
$$

But, since the autocorrelation function is symmetric and positive, and the process's mean is a real-valued quantity, the sign of μ depends on additional context or conventions.

Typically, in such cases, the mean is taken as the positive root:

$$
\boxed{
\text{Mean} \, \mu = \sqrt{3}
}
$$

Conclusion

Summary of Findings

- The autocorrelation function RXX(τ) is given by $3 + 9e^{-|\tau|}$.
- At τ=0, RXX(0) = 12, which equals $\text{Var}(X) + \mu^2$.
- As |τ| approaches infinity, RXX(τ) approaches 3, which is μ^2.

Final Computation of the Mean

From the asymptotic behavior:

$$
\mu^2 = 3 \Rightarrow \mu = \pm \sqrt{3}
$$

Considering typical conventions and the positivity of the autocorrelation at zero, the mean is:

$$
\boxed{
\text{Mean} \, \mu = \sqrt{3}
}
$$

Therefore, the mean of the process X(t) is $\boxed{\sqrt{3}}$.

Additional Remarks

- The sign of the mean can be positive or negative depending on the context; here, the positive root is generally taken unless specified otherwise.
- The autocorrelation function's form suggests the process might be modeled as a stationary Gaussian process with an exponential autocorrelation structure.
- Knowledge of the autocorrelation function allows for inferences about the underlying process's statistical properties, such as mean, variance, and correlation decay rate.

References and Further Reading

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This comprehensive analysis demonstrates how the autocorrelation function provides critical insights into the statistical parameters of a random process, specifically its mean, variance, and correlation structure.

Frequently Asked Questions

What does the autocorrelation function RXX(τ) represent in a random process?

The autocorrelation function RXX(τ) measures the correlation between the values of the random process X(t) at times t and t+τ, indicating how the process values are related over time.

Given RXX(τ) = 3 + 9e^(-|τ|), how can we determine the mean of the process X(t)?

The mean of the process X(t) is given by RXX(0), which is the autocorrelation at zero lag, so the mean is √(RXX(0)) if the process is zero-mean, but typically, the mean is directly obtained from the autocorrelation at τ=0, which in this case is 3.

What is the significance of the term 3 in the autocorrelation function RXX(τ) = 3 + 9e^(-|τ|)?

The term 3 represents the constant part of the autocorrelation at zero lag, which is related to the variance of the process if the process has zero mean; it indicates the level of correlation when τ=0.

How do you find the variance of the process X(t) from its autocorrelation function?

The variance is obtained by evaluating RXX(0) minus the square of the mean: Variance = RXX(0) - [E[X(t)]]^2. If the mean is zero, variance equals RXX(0).

Is the process described by RXX(τ) = 3 + 9e^(-|τ|) stationary? Why?

Yes, the process is wide-sense stationary because its autocorrelation function depends only on the time lag τ and not on the absolute time t.

What does the exponential term 9e^(-|τ|) tell us about the correlation of the process over time?

The exponential term indicates that the autocorrelation decays exponentially with increasing |τ|, showing the process has a finite correlation time and is correlated more strongly at shorter lags.

How can the autocorrelation function be used to determine the power spectral density of the process?

The power spectral density (PSD) is the Fourier transform of the autocorrelation function RXX(τ). Applying Fourier transform to RXX(τ) yields the PSD, which describes the distribution of power over frequency.

If RXX(τ) = 3 + 9e^(-|τ|), what is the value of the autocorrelation at large |τ|?

As |τ| approaches infinity, e^(-|τ|) approaches zero, so RXX(τ) approaches 3, indicating the residual correlation level at large time separations.

Can the mean of the process X(t) be directly obtained from the autocorrelation function provided?

Not directly, unless the process is zero-mean; the autocorrelation at $\tau=0$ gives the total power (variance plus squared mean), so additional information about the mean is needed to separate it from the autocorrelation.

Additional Resources

Understanding the Autocorrelation Function of a Random Process X(t): A Comprehensive Guide

When analyzing random processes in fields such as signal processing, communications, and control systems, the autocorrelation function of a random process X(t) plays a pivotal role in characterizing its statistical properties. Given the autocorrelation function \(R_{XX}(\tau) = 3 + 9e^{-\frac{|\tau|}{\tau_0}} \), a fundamental question arises: What is the mean of the process X(t)? This guide aims to demystify this problem by exploring the core concepts of autocorrelation functions, their relationship with the mean and variance, and the step-by-step approach to extracting the mean value from the given autocorrelation expression.

Introduction to the Autocorrelation Function

The autocorrelation function \(R_{XX}(\tau) \) of a stationary random process \(X(t) \) is defined as:

\[
R_{XX}(\tau) = E[X(t) X(t + \tau)]
\]

where \(E[\cdot] \) denotes the expectation operator, and \(\tau \) is the time lag. It measures how similar the process is to itself at different time shifts, providing insights into the correlation structure over time.

Properties of the Autocorrelation Function

- Symmetry: \(R_{XX}(\tau) = R_{XX}(-\tau) \).
- Maximum at zero lag: \(R_{XX}(0) \) signifies the total power or variance of the process.
- Relation to Mean and Variance: For a wide-sense stationary process, the autocorrelation function is related to the mean \(\mu \) and variance \(\sigma^2 \) as:

\[
R_{XX}(0) = E[X(t)^2] = \sigma^2 + \mu^2
\]

Deciphering the Given Autocorrelation Function

Given:

$$
R_{XX}(\tau) = 3 + 9 e^{-\frac{|\tau|}{\tau_0}}
$$

where τ_0 is a positive constant parameter (often called the correlation time or decay constant). Our goal is to find the mean $\mu = E[X(t)]$ of the process.

Connecting Autocorrelation to the Mean

Key Relationship

For second-order stationary processes:

$$
R_{XX}(\tau) = E[X(t) X(t + \tau)] = E[X(t)] E[X(t + \tau)] + \text{Cov}(X(t), X(t + \tau))
$$

Since the process is stationary, $E[X(t)] = E[X(t + \tau)] = \mu$. Therefore:

$$
R_{XX}(\tau) = \mu^2 + \text{Cov}(X(t), X(t + \tau))
$$

At zero lag ($\tau = 0$):

$$
R_{XX}(0) = \mu^2 + \sigma^2
$$

where σ^2 is the variance of $X(t)$.

Implication for the Given Function

Given:

$$
R_{XX}(0) = 3 + 9 e^{0} = 3 + 9 = 12
$$

since $e^{0} = 1$.

Therefore:

$$
R_{XX}(0) = 12
$$

which leads to:

$$
12 = \mu^2 + \sigma^2
$$

This is a key equation linking the mean and variance.

Determining the Mean μ

To find μ, we need additional information or assumptions. Typically, if the autocorrelation function corresponds to a process with zero mean (such as zero-mean noise), then $\mu = 0$. However, in the absence of such an assumption, we need to analyze further.

Common Approach: Stationary Processes with Known Autocorrelation

In many practical scenarios, the autocorrelation function is expressed as:

$$
R_{XX}(\tau) = \text{constant} + \text{decaying exponential term}
$$

The constant term often equals μ^2, the square of the mean, especially for processes where the autocorrelation function is separated into a mean component and a fluctuating component.

Thus, if we interpret:

$$
R_{XX}(\tau) = \text{constant} + \text{exponential decay}
$$

then the constant term corresponds to μ^2.

In our case:

$$
R_{XX}(\tau) = 3 + 9 e^{-\frac{|\tau|}{\tau_0}}
$$

- The constant term is 3.
- The decaying exponential term is $9 e^{-\frac{|\tau|}{\tau_0}}$.

Therefore, the autocorrelation function's value at large $|\tau|$ (i.e., as $|\tau| \to \infty$) is:

$$
\lim_{|\tau|\to \infty} R_{XX}(\tau) = 3
$$

since $e^{-\frac{|\tau|}{\tau_0}} \to 0$.

Interpretation

- The autocorrelation function approaches 3 at large lags, indicating a persistent component possibly related to the square of the mean.
- The initial value at $\tau = 0$ is 12, which combines the contributions of the mean and variance.

Deriving the Mean from the Autocorrelation Function

Given the above, the most common and consistent interpretation is:

$$
\boxed{
\mu^2 = \text{lim}_{|\tau|\to \infty} R_{XX}(\tau) = 3
}
$$

which implies:

$$
\mu = \pm \sqrt{3}
$$

Since the autocorrelation function involves the square of the mean, the sign of the mean cannot be directly determined from autocorrelation alone. However, in many contexts, the mean is taken as positive unless specified otherwise.

Summary of the Key Findings

- The autocorrelation function approaches 3 at large $|\tau|$, indicating the square of the mean:

$$
\boxed{
\mu^2 = 3 \Rightarrow \mu = \pm \sqrt{3}
}
$$

- The autocorrelation at zero lag:

$$
R_{XX}(0) = 12
$$

- The variance:

$$
\sigma^2 = R_{XX}(0) - \mu^2 = 12 - 3 = 9
$$

which confirms the process has a variance of 9.

Final Conclusion: The Mean of $X(t)$

Based on the analysis, the mean of the process $X(t)$ is:

$$
\boxed{
\boxed{
\mu = \pm \sqrt{3}
}
}
$$

If the context suggests a positive mean, then:

$$
\boxed{
\mu = \sqrt{3} \approx 1.732
}
$$

Otherwise, the negative root $-\sqrt{3}$ is also mathematically valid.

Additional Insights and Practical Considerations

Significance of the Autocorrelation Function

Understanding the autocorrelation function helps in designing filters, predicting future values, and analyzing the power spectral density of the process.

Power Spectral Density (PSD)

The Fourier transform of $R_{XX}(\tau)$ yields the PSD, providing frequency domain insights. For the given autocorrelation:

$$
S_{XX}(\omega) = \mathcal{F}\{ R_{XX}(\tau) \}
$$

which involves the transform of a constant plus an exponential decay.

Assumptions in the Analysis

- The process is second-order stationary.
- The autocorrelation function is correctly specified.
- The process has a well-defined mean and variance.

Summary in Bullet Points

- The autocorrelation function $R_{XX}(\tau) = 3 + 9 e^{-\frac{|\tau|}{\tau_0}}$ indicates a process with a constant component and an exponentially decaying component.
- The value of $R_{XX}(\tau)$ at large $|\tau|$ (as $|\tau| \to \infty$) provides the square of the mean:

$$
\mu^2 = 3 \Rightarrow \mu = \pm \sqrt{3}
$$

- The autocorrelation at zero lag yields the total power:

$$
R_{XX}(0) = 12
$$

- The variance of $X(t)$:

$$
\sigma^2 = R_{XX}(0) - \mu^2 = 9
$$

- The process likely has a mean $\boxed{\sqrt{3}}$ or $-\sqrt{3}$, depending on additional context.

Final Remarks

Understanding the autocorrelation function is fundamental in stochastic process analysis. By examining its structure, limits, and relationships to statistical moments, we can extract vital parameters such as the mean and variance. The given autocorrelation function demonstrates classic features of a process with a persistent mean component and a correlation decay, a common scenario in physical and engineering systems.

This comprehensive guide aimed to clarify

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