# The Function Y1=e^(3x) Is A Solution Of Y''-6y'+9y=0. Find A Second Linearly Independent Solution Y2

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# Introduction to the Differential Equation

Understanding second-order linear differential equations is fundamental in various fields such as physics, engineering, and mathematics. The given differential equation:

$$ Y'' - 6Y' + 9Y = 0 $$

is a homogeneous linear differential equation with constant coefficients. Solving such equations typically involves finding the general solution, which is a linear combination of two linearly independent solutions.

In this context, the problem states that:

$$ Y_1 = e^{3x} $$

is a solution. The task is to find another solution $Y_2$ that is linearly independent of $Y_1$. This process involves understanding characteristic equations, roots, and methods like reduction of order to find the second solution.

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# Understanding the Characteristic Equation

## Deriving the Characteristic Equation

For linear differential equations with constant coefficients, the standard approach is to assume solutions of the form:

$$ Y = e^{rx} $$

where $r$ is a constant to be determined. Substituting into the differential equation:

$$ r^2 e^{rx} - 6 r e^{rx} + 9 e^{rx} = 0 $$

Dividing through by $e^{rx}$ (which is never zero):

$$ r^2 - 6r + 9 = 0 $$

This quadratic equation is known as the characteristic (or auxiliary) equation:

$$ r^2 - 6r + 9 = 0 $$

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# Solving the Characteristic Equation

## Finding Roots of the Quadratic

The quadratic equation:

$$ r^2 - 6r + 9 = 0 $$

can be factored or solved using the quadratic formula:

$$ r = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 9}}{2} $$

Calculating the discriminant:

$$ \Delta = 36 - 36 = 0 $$

Since the discriminant is zero, this quadratic has a repeated root:

$$ r = \frac{6}{2} = 3 $$

Therefore, the characteristic equation has a double root:

$$ r = 3 $$

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# Implications of Repeated Roots in Differential Equations

# General Solution for Repeated Roots

When the characteristic equation has a repeated root $r$, the general solution to the differential equation takes the form:

$$y(x) = (A + Bx) e^{rx}$$

where $A$ and $B$ are arbitrary constants.

Given that:

$$r = 3$$

the first solution:

$$Y_1 = e^{3x}$$

corresponds to the root $r = 3$.

To find the second linearly independent solution $Y_2$, we employ the method of reduction of order.

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# Method of Reduction of Order

## Overview of Reduction of Order

The reduction of order technique is used to find a second solution $Y_2$ when one solution $Y_1$ is known. It involves assuming:

$$Y_2 = v(x) Y_1$$

where $v(x)$ is an unknown function to be determined.

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## Applying Reduction of Order

Given $Y_1 = e^{3x}$, assume:

$$Y_2 = v(x) e^{3x}$$

Compute derivatives:

$$ Y_2' = v' e^{3x} + 3 v e^{3x} $$
$$ Y_2'' = v'' e^{3x} + 6 v' e^{3x} + 9 v e^{3x} $$

Substitute into the original differential equation:

$$ Y'' - 6Y' + 9Y = 0 $$

which becomes:

$$ (v'' e^{3x} + 6 v' e^{3x} + 9 v e^{3x}) - 6 (v' e^{3x} + 3 v e^{3x}) + 9 v e^{3x} = 0 $$

Simplify each term:

$$ v'' e^{3x} + 6 v' e^{3x} + 9 v e^{3x} - 6 v' e^{3x} - 18 v e^{3x} + 9 v e^{3x} = 0 $$

Combine like terms:

$$ v'' e^{3x} + (6 v' e^{3x} - 6 v' e^{3x}) + (9 v e^{3x} - 18 v e^{3x} + 9 v e^{3x}) = 0 $$

Simplify further:

$$ v'' e^{3x} + 0 + (0) = 0 $$

which simplifies to:

$$ v'' e^{3x} = 0 $$

Dividing both sides by $ e^{3x} $:

$$ v'' = 0 $$

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# Solving the Reduced Equation

The differential equation:

$$ v'' = 0 $$

has the general solution:

$$ v(x) = C_1 x + C_2 $$

where $ C_1, C_2 $ are arbitrary constants.

Since $ Y_2 = v(x) e^{3x} $, the second solution is:

$$ Y_2 = v(x) e^{3x} = (C_1 x + C_2) e^{3x} $$

To find a linearly independent solution, choose $C_1 = 1$ and $C_2 = 0$:

$$ Y_2 = x e^{3x} $$

Thus, the second linearly independent solution is:

$$ \boxed{Y_2 = x e^{3x}} $$

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# Final General Solution

The general solution to the differential equation:

$$ Y'' - 6Y' + 9Y = 0 $$

which accounts for the repeated root, is:

$$ Y(x) = C_1 e^{3x} + C_2 x e^{3x} $$

where $C_1, C_2$ are arbitrary constants.

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# Summary and Key Takeaways

- The characteristic equation for the differential equation has a repeated root $r = 3$.
- The first known solution is $Y_1 = e^{3x}$.
- Using reduction of order, the second linearly independent solution is $Y_2 = x e^{3x}$.
- The general solution combines both solutions: $Y(x) = C_1 e^{3x} + C_2 x e^{3x}$.

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# Implications and Applications

Understanding how to find solutions to differential equations with repeated roots is essential for solving real-world problems. These solutions are fundamental in systems where resonance occurs, such as in mechanical vibrations, electrical circuits, and wave phenomena.

The method of reduction of order provides a systematic approach to obtaining a second solution when one solution is known, especially in cases of repeated roots, ensuring the completeness of the solution space.

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# Conclusion

In conclusion, starting from the given differential equation and the known solution $Y_1 = e^{3x}$, we employed the characteristic equation and the reduction of order method to derive the second linearly independent solution $Y_2 = x e^{3x}$. Together, these solutions form the basis for the general solution to the differential equation, enabling the analysis of various physical and mathematical systems modeled by such equations.

# Frequently Asked Questions

## What is the differential equation satisfied by the function $y_1 = e^{3x}$?

The differential equation is $y'' - 6y' + 9y = 0$.

## How do you verify that $y_1 = e^{3x}$ is a solution to the differential equation?

Substitute $y_1 = e^{3x}$ into the differential equation and verify that it satisfies the equation by computing $y'$ and $y''$.

## What is the characteristic equation associated with the differential equation $y'' - 6y' + 9y = 0$?

The characteristic equation is $r^2 - 6r + 9 = 0$.

## What are the roots of the characteristic equation $r^2 - 6r + 9 = 0$?

The roots are $r = 3$ (a repeated root).

## Given that $y_1 = e^{3x}$ is a solution, how do you find a second linearly independent solution $y_2$?

Since $r = 3$ is a repeated root, $y_2$ can be found using the method of reduction of order, resulting in $y_2 = x e^{3x}$.

## What is the second linearly independent solution $y_2$ for the differential equation?

The second solution is $y_2 = x e^{3x}$.

## Why is $y_2 = x e^{3x}$ linearly independent from $y_1 = e^{3x}$?

Because $y_2$ involves a factor of x, making it linearly independent from $y_1$, which does not include x.

## What is the general solution to the differential equation $y'' - 6y' + 9y = 0$?

The general solution is $y(x) = C_1 e^{3x} + C_2 x e^{3x}$, where $C_1$ and $C_2$ are arbitrary constants.

## How does the presence of a repeated root affect the form of the solutions to a second-order linear differential equation?

A repeated root r results in solutions of the form $e^{rx}$ and $x e^{rx}$, ensuring linear independence.

## What method is used to find the second solution when the roots of the characteristic equation are repeated?

The method of reduction of order is used to find the second linearly independent solution.

# Additional Resources

### Understanding the Differential Equation $y'' - 6y' + 9y = 0$ and Its Solutions

Mathematics, particularly differential equations, often provides profound insights into the natural and engineered systems around us. Among the foundational concepts in this field is the idea of solving linear differential equations with constant coefficients. These equations frequently appear in physics, engineering, and other scientific disciplines to model dynamic systems, from oscillating springs to electrical circuits. A pivotal aspect of solving such equations involves identifying solutions and understanding their linear independence, which ensures the generality of the solutions.

In this article, we explore a specific second-order homogeneous linear differential equation:

$$
y'' - 6y' + 9y = 0,
$$

and analyze a particular solution:

$$
Y_1 = e^{3x},
$$

delving into how it fits into the broader solution space and how to determine a second linearly independent solution, $Y_2$. Our discussion will encompass the theoretical background, detailed solution derivations, and the significance of linear independence in differential equations.

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## The Nature of the Differential Equation: Homogeneity and Linearity

Understanding the structure of the differential equation

$$
y'' - 6y' + 9y = 0
$$

begins with recognizing that it is:

- Homogeneous: The right-hand side equals zero, meaning all solutions satisfy the equation without external forcing functions.
- Linear: The dependent variable $y$ and its derivatives appear to the first power and are not multiplied together; this property simplifies solutions and allows superposition.

These traits allow the use of classic methods such as characteristic equations to find general solutions.

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## Characteristic Equation and Its Roots

Transforming the differential equation into an algebraic problem involves assuming solutions of the form:

$$
y = e^{rx},
$$

where $r$ is a constant to be determined.

Substituting $y = e^{rx}$, $y' = r e^{rx}$, and $y'' = r^2 e^{rx}$, into the differential equation yields:

$$
r^2 e^{rx} - 6 r e^{rx} + 9 e^{rx} = 0.
$$

Dividing through by $e^{rx}$ (which is never zero), we obtain the characteristic equation:

$$
r^2 - 6r + 9 = 0.
$$

This quadratic simplifies to:

$$
(r - 3)^2 = 0,
$$

indicating a repeated root:

$$
r = 3,
$$

with multiplicity two.

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# Implications of Repeated Roots on Solutions

In the theory of differential equations, the nature of characteristic roots profoundly influences the form of solutions:

- Distinct real roots: The general solution is a linear combination of exponential functions.
- Repeated roots: The general solution includes terms multiplied by powers of $x$ to account for multiplicity.

Given $r = 3$ is a repeated root, the general solution to the differential equation takes the form:

$$

$$
y(x) = C_1 e^{3x} + C_2 x e^{3x},
$$

where $C_1$ and $C_2$ are arbitrary constants, and the two solutions:

$$
Y_1 = e^{3x},
$$
$$
Y_2 = x e^{3x},
$$

are linearly independent.

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## Verifying the Given Solution $Y_1 = e^{3x}$

The problem states that $Y_1 = e^{3x}$ is a solution to the differential equation:

$$
y'' - 6 y' + 9 y = 0.
$$

Let's verify this explicitly:

- Compute the derivatives:

$$
Y_1' = 3 e^{3x},
$$
$$
Y_1'' = 9 e^{3x}.
$$

- Substitute into the differential equation:

$$
9 e^{3x} - 6 (3 e^{3x}) + 9 e^{3x} = 9 e^{3x} - 18 e^{3x} + 9 e^{3x} = (9 - 18 + 9) e^{3x} = 0.
$$

Since the expression simplifies to zero, $Y_1 = e^{3x}$ is indeed a solution.

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# Finding a Second Linearly Independent Solution $Y_2$

Given that $e^{3x}$ is a solution associated with a repeated root, the general theory predicts that a second, linearly independent solution must be of the form:

$$
Y_2 = x e^{3x}.
$$

This form arises from the method of reduction of order, which systematically constructs solutions when one solution is known.

Method of Reduction of Order

Suppose $y_1 = e^{3x}$ is known. We seek a second solution $y_2$ in the form:

$$
y_2 = v(x) y_1 = v(x) e^{3x}.
$$

Differentiating:

$$
y_2' = v' e^{3x} + v \cdot 3 e^{3x},
$$
$$
y_2'' = v'' e^{3x} + 2 v' \cdot 3 e^{3x} + v \cdot 9 e^{3x}.
$$

Substituting into the original differential equation:

$$
(y_2)'' - 6 y_2' + 9 y_2 = 0,
$$

and simplifying, leads to a differential equation for $v(x)$:

$$
v'' e^{3x} + 6 v' e^{3x} + 9 v e^{3x} - 6 v' e^{3x} - 18 v e^{3x} + 9 v e^{3x} = 0.
$$

Collecting terms:

$$
v'' e^{3x} + (6 v' e^{3x} - 6 v' e^{3x}) + (9 v e^{3x} - 18 v e^{3x} + 9 v e^{3x}) = 0,
$$

which simplifies to:

$$
v'' e^{3x} = 0,
$$
or

$$
v'' = 0.
$$

Integrating twice:

$$
v' = A,
$$
$$
v = A x + B,
$$

where $A, B$ are constants. Since only the term involving $v$ matters for the second independent solution, and the solution must be linearly independent of $e^{3x}$, we set $A = 1$, $B = 0$, leading to:

$$
v(x) = x.
$$

Thus,

$$
Y_2 = v(x) y_1 = x e^{3x}.
$$

Conclusion: The second linearly independent solution is

$$
Y_2 = x e^{3x}.
$$

---

# The General Solution and Its Significance

The general solution to the differential equation combines both independent solutions:

$$
\boxed{
y(x) = C_1 e^{3x} + C_2 x e^{3x}.
}
$$

The constants $C_1$ and $C_2$ reflect initial conditions or boundary values, shaping specific solutions relevant to physical or engineering problems.

Why is linear independence important?

- It ensures the solution space is two-dimensional, matching the order of the differential equation.
- It guarantees that any solution can be expressed as a linear combination of these two solutions.
- It allows the application of initial conditions to solve for the constants uniquely.

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# Applications and Broader Context

Understanding solutions like $e^{3x}$ and $x e^{3x}$ is crucial in various contexts:

- Mechanical systems: Damped or undamped oscillators with repeated roots in their characteristic equations.
- Electrical circuits: RLC circuits where the differential equations exhibit repeated roots.
- Population models: Certain growth models with constant coefficients can reduce to similar equations.
- Control systems: Stability analysis often involves solving characteristic equations with repeated roots.

In real-world applications, recognizing the form of solutions and their independence ensures accurate modeling, control, and prediction.

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# Conclusion: The Power of Mathematical Rigor in Differential Equations

The exploration of the differential equation $y'' - 6 y' + 9 y = 0$ exemplifies how fundamental mathematical principles underpin the analysis of complex systems. Starting from the characteristic equation, identifying the root multiplicity, and applying reduction of order, we derived the complete set of solutions. The explicit verification of $Y_1 = e^{3x}$ as a solution and the methodical derivation of $Y_2 = x e^{3x}$ underscore the elegance and consistency of linear differential equations.

Appreciating the linear independence of solution functions is not merely a mathematical curiosity; it is essential for constructing the general solution, interpreting physical phenomena, and solving practical problems

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