The IVP Sin(t)dx/dt+cos(t) Dx/dt+sin(t)x=tan(t) X(1.25)=4dx/dt|1.25 = 1has A Unique Solution Defined

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In the realm of differential equations, understanding the existence and uniqueness of solutions to initial value problems (IVPs) is fundamental. The specific IVP given by the equation sin(t) dx/dt + cos(t) Dx/dt + sin(t) x = tan(t), with initial conditions X(1.25) = 4 and dx/dt at t=1.25 equal to 1, is a classic example illustrating how mathematical theory ensures that a unique solution exists under certain conditions. This article explores this particular IVP in detail, examining the underlying concepts, methods for solving, and the conditions that guarantee the solution's uniqueness. Whether you're a student, educator, or researcher, understanding this problem enhances your grasp of differential equations and their applications.

Understanding the Initial Value Problem (IVP)

What Is an IVP?

An initial value problem involves a differential equation along with specified initial conditions. It seeks a function x(t) that satisfies the differential equation and meets the initial conditions at a particular point $t = t_0$. Formally, an IVP can be written as:

$$
\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0
$$

The question of whether a solution exists and whether it is unique is central to the study of differential equations.

The Specific IVP Under Consideration

The problem we're analyzing is:

$$
\sin(t) \frac{dx}{dt} + \cos(t) \frac{dX}{dt} + \sin(t) x = \tan(t)
$$

with initial conditions:

$$
X(1.25) = 4, \quad \frac{dx}{dt}\bigg|_{t=1.25} = 1
$$

At first glance, this looks complex, but by carefully examining its structure, we can determine the existence and uniqueness of solutions.

Rewriting the Differential Equation for Clarity

Identifying the Type of Equation

The given differential equation combines derivatives of x and X, suggesting it may involve coupled or higher-order relations. To analyze it effectively, we need to clarify the relationship between the variables and derivatives involved.

Note that the notation suggests that $X(t)$ is a function related to $x(t)$, possibly representing an auxiliary function or a typo. For clarity, assume that the primary unknown function is $x(t)$, and the equation involves its derivatives, as well as an auxiliary derivative term.

Alternatively, if the notation indicates a second variable or a typo, further clarification is necessary. However, for the purpose of this discussion, assume the equation can be expressed as:

$$
\sin(t) \frac{dx}{dt} + \cos(t) \frac{dX}{dt} + \sin(t) x = \tan(t)
$$

If $X(t)$ is the same as $x(t)$, or if the derivatives of X are related to x, then the equation simplifies accordingly.

Expressing the Equation as a First-Order ODE

Suppose the goal is to analyze the differential equation for $x(t)$. If the derivatives of X are related to x, perhaps as part of a system, then the problem might be coupled. Alternatively, if the notation is complex, focusing on the main variable $x(t)$ and the initial condition is more straightforward.

Given the initial conditions and the form of the equation, we can proceed by considering the equation as a first-order linear differential equation with variable coefficients.

Determining Existence and Uniqueness of Solutions

Conditions for Uniqueness: The Picard-Lindelöf Theorem

The Picard-Lindelöf theorem states that if a differential equation can be expressed as:

$$
\frac{dx}{dt} = f(t, x)
$$

and if $f(t, x)$ and its partial derivative with respect to x are continuous in a neighborhood around the initial conditions, then there exists a unique solution passing through the initial point.

Applying this to our problem, the key is to verify the continuity of the coefficient functions involved and ensure they satisfy the necessary conditions.

Applying to Our Differential Equation

Rearranged, the differential equation can be written as:

$$
\sin(t) \frac{dx}{dt} + \sin(t) x = \tan(t) - \cos(t) \frac{dX}{dt}
$$

If we assume the derivatives of X are known or related, and focus on the equation involving only x(t), then the structure resembles a linear first-order ODE:

$$
\frac{dx}{dt} + p(t) x = q(t)
$$

where:

$$
p(t) = \frac{\sin(t)}{\sin(t)} = 1
$$

$$
q(t) = \frac{\tan(t) - \cos(t) \frac{dX}{dt}}{\sin(t)}
$$

Provided that $p(t)$ and $q(t)$ are continuous around $t=1.25$, and the initial conditions are specified at that point, then by Picard-Lindelöf, a unique solution exists.

Verifying the Conditions at t=1.25

Continuity of Coefficient Functions

- $\sin(t)$ is continuous except at points where $\sin(t) = 0$, i.e., multiples of π. Since 1.25 is not a multiple of π, $\sin(1.25) \neq 0$, ensuring the coefficient functions are well-defined and continuous at t=1.25.
- $\tan(t)$ is continuous where $\cos(t) \neq 0$. At t=1.25, $\cos(1.25) \neq 0$, so $\tan(t)$ is continuous there.
- The derivative $\frac{dX}{dt}$ at t=1.25 is given as 1, which helps define $q(t)$.

Since all these functions are continuous at t=1.25, the conditions for the existence and uniqueness of the solution are satisfied.

Solving the Differential Equation

Method: Integrating Factor

For a linear first-order differential equation:

$$
\frac{dx}{dt} + p(t) x = q(t)
$$

the integrating factor is:

$$
\mu(t) = e^{\int p(t) dt}
$$

Given that $p(t) = 1$, then:

$$
\mu(t) = e^{t}
$$

Multiplying through the differential equation by $\mu(t)$:

$$
e^{t} \frac{dx}{dt} + e^{t} x = e^{t} q(t)
$$

which simplifies to:

$$
\frac{d}{dt}\left(e^{t} x \right) = e^{t} q(t)
$$

Integrating both sides from the initial point $t = 1.25$:

$$
e^{t} x(t) = e^{1.25} x(1.25) + \int_{1.25}^{t} e^{s} q(s) ds
$$

Applying the initial condition $x(1.25) = 4$:

$$
x(t) = e^{-t} \left[e^{1.25} \times 4 + \int_{1.25}^{t} e^{s} q(s) ds \right]
$$

The explicit solution depends on the integral of $q(s)$, which in turn depends on $\frac{dX}{dt}$. If $\frac{dX}{dt}$ is known or can be expressed in terms of x(t), then the solution can be fully determined.

Ensuring the Solution Is Unique and Well-Defined

Key Points Summary

- The coefficients involved are continuous around t=1.25.
- The initial conditions specify both $X(1.25) = 4$ and $\frac{dx}{dt}|_{t=1.25} = 1$.
- The differential equation can be written in a standard linear form, allowing the use of the integrating factor method.
- The conditions satisfy the hypotheses of the Picard-Lindelöf theorem, guaranteeing a unique solution.

Implications of Uniqueness

The guarantee of a unique solution means that for the specified initial conditions and the given differential equation, there exists exactly one function x(t) that satisfies the equation in some interval around t=1.25. This result is fundamental in the theory of differential equations because it ensures predictability and stability of solutions under specified initial conditions.

Applications and Significance in Mathematics and Engineering

Practical Applications

- Modeling physical systems where initial states are known, such as in mechanics, electronics, or biological systems.
- Numerical simulations that rely on unique solutions to ensure consistent results.
- Engineering design, where initial conditions determine system behavior.

Importance in Teaching and

Frequently Asked Questions

What type of differential equation is given by IVP Sin(t) dx/dt + cos(t) dx/dt + sin(t) x = tan(t)?

It is a first-order linear differential equation with variable coefficients.

How do you determine whether the initial value problem has a unique solution?

By verifying that the functions multiplying dx/dt and x are continuous

near the initial point and that the initial conditions are specified, ensuring the existence and uniqueness theorem applies.

What is the significance of the initial condition $X(1.25) = 4$ and $dx/dt|_{t=1.25} = 1$?

These initial conditions specify the value of the function and its derivative at t=1.25, which are essential for finding the particular solution to the IVP.

How can the differential equation be simplified to identify the solution method?

By rewriting it in standard linear form and identifying the integrating factor or using substitution methods suitable for variable coefficients.

What role does the tan(t) right-hand side play in solving this IVP?

It serves as the nonhomogeneous part of the differential equation, influencing the particular solution that must be added to the homogeneous solution.

Why does the continuity of coefficient functions matter in this IVP?

Because continuity ensures the conditions for the existence and uniqueness of solutions are met, according to the Picard-Lindelöf theorem.

Can this IVP be solved analytically, or is numerical methods preferred?

It can potentially be solved analytically using integrating factors and known functions, but numerical methods may be used for approximation if the solution form is complicated.

What is the significance of the statement 'has a unique solution' in the context of this IVP?

It indicates that, given the initial conditions, there exists exactly one solution that satisfies both the differential equation and the initial conditions.

How do the initial conditions at t=1.25 influence the form of the solution?

They specify specific values for the solution and its derivative at t=1.25, which are used to determine the arbitrary constants in the general solution.

What are the key steps to solving this IVP explicitly?

Identify the standard form, find the integrating factor if applicable, solve the homogeneous equation, find a particular solution, apply initial conditions to determine constants, and write the explicit solution.

Additional Resources

The IVP Sin(t) dx/dt + cos(t) x' + sin(t) x = tan(t); X(1.25) = 4; dx/dt | 1.25 = 1 has a Unique Solution Defined

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Introduction

In the vast landscape of differential equations, initial value problems (IVPs) occupy a central position, especially those that describe dynamic systems in physics, engineering, and applied mathematics. The particular IVP under consideration involves a first-order linear differential equation with variable coefficients:

$$
\sin(t) \frac{dx}{dt} + \cos(t) x + \sin(t) x = \tan(t),
$$

accompanied by initial conditions:

$$
X(1.25) = 4, \quad \text{and} \quad \frac{dx}{dt}\bigg|_{t=1.25} = 1.
$$

This problem is notable not only for the specific functions involved but also for the question of whether a unique solution exists given these initial conditions. Understanding the existence and uniqueness of solutions to such IVPs is fundamental to ensuring the reliability of models that depend on differential equations.

In this article, we explore the structure of this differential equation, analyze the conditions under which a unique solution exists, and interpret the implications for applied mathematics and related disciplines.

Understanding the Differential Equation

Formulation and Simplification

The differential equation is:

$$
\sin(t) \frac{dx}{dt} + [\cos(t) + \sin(t)] x = \tan(t).
$$

For clarity, rewrite it in a standard linear form:

$$
\frac{dx}{dt} + P(t) x = Q(t),
$$

by dividing both sides by $\sin(t)$, assuming $\sin(t) \neq 0$:

$$
\frac{dx}{dt} + \frac{\cos(t) + \sin(t)}{\sin(t)} x = \frac{\tan(t)}{\sin(t)}.
$$

Simplify the coefficients:

- The integrating factor and the coefficients become:

$$
P(t) = \frac{\cos(t) + \sin(t)}{\sin(t)} = \cot(t) + 1,
$$

$$
Q(t) = \frac{\tan(t)}{\sin(t)}.
$$

Recognizing that:

$$
\cot(t) = \frac{\cos(t)}{\sin(t)},
$$

$$
Q(t) = \frac{\tan(t)}{\sin(t)} = \frac{\frac{\sin(t)}{\cos(t)}}{\sin(t)} = \frac{1}{\cos(t)}.
$$

Thus, the differential equation simplifies to:

$$
\frac{dx}{dt} + (\cot(t) + 1) x = \frac{1}{\cos(t)}.
$$

This form is valid for t in an interval where $\sin(t) \neq 0$ and $\cos(t) \neq 0$.

Existence and Uniqueness of the Solution

Theoretical Background

The fundamental theorem for first-order linear differential equations states that:

> Given a linear differential equation $\frac{dx}{dt} + P(t) x = Q(t)$, if both $P(t)$ and $Q(t)$ are continuous on an interval I, then for any

initial condition $x(t_0) = x_0$ where $t_0 \in I$, there exists a unique solution $x(t)$ defined on I.

Therefore, the crux of the solution's existence and uniqueness hinges on the continuity of $P(t)$ and $Q(t)$ on the interval of interest, particularly around the initial point $t=1.25$.

Analyzing the Coefficients and Initial Conditions

- The functions $\cot(t)$ and $\frac{1}{\cos(t)}$ are both continuous where $\sin(t) \neq 0$ and $\cos(t) \neq 0$. Since $\cot(t)$ is discontinuous at points where $\sin(t)=0$, i.e., at integer multiples of π, the interval of validity excludes these points.

- At $t=1.25$, which is approximately 1.25 radians (~71.6 degrees), neither $\sin(1.25)$ nor $\cos(1.25)$ is zero, so the coefficients are well-defined and continuous around $t=1.25$.

- The initial condition $X(1.25)=4$ provides a specific starting point, and the derivative condition $dx/dt|_{t=1.25}=1$ offers additional information, confirming the initial slope.

Hence, within an interval containing $t=1.25$ where $\sin(t)$ and $\cos(t)$ remain non-zero, the conditions for the existence and uniqueness theorem are satisfied.

Interval of Validity

To ensure a unique solution is defined, the interval must:

- Contain the point $t=1.25$.
- Exclude points where $\sin(t)=0$ (i.e., $t = n\pi$, where n is an integer).
- The interval should be maximal where the coefficients $P(t)$ and $Q(t)$ are continuous.

For instance, a typical interval might be:

$$
(0.5, 2.0),
$$

excluding points like $t=\pi \approx 3.1416$, or more precisely, an interval symmetric around 1.25 that avoids crossing any singularities.

Solving the Differential Equation

Integrating Factor Method

Given the standard form:

$$
\frac{dx}{dt} + P(t) x = Q(t),
$$

where:

$$
P(t) = \cot(t) + 1,
$$

$$
Q(t) = \frac{1}{\cos(t)}.
$$

The integrating factor $\mu(t)$ is:

$$
\mu(t) = e^{\int P(t) dt} = e^{\int (\cot(t) + 1) dt} = e^{\int \cot(t) dt + \int 1 dt}.
$$

Compute each integral:

- $\int \cot(t) dt = \ln |\sin(t)| + C$,
- $\int 1 dt = t + C$.

Thus, the integrating factor:

$$
\mu(t) = e^{\ln |\sin(t)| + t} = |\sin(t)| e^{t}.
$$

Since $\sin(t) > 0$ for t in $(0, \pi)$, we can write:

$$
\mu(t) = \sin(t) e^{t}.
$$

The general solution:

$$
x(t) = \frac{1}{\mu(t)} \left(\int \mu(t) Q(t) dt + C \right).
$$

Substitute:

$$
x(t) = \frac{1}{\sin(t) e^{t}} \left(\int \sin(t) e^{t} \cdot \frac{1}{\cos(t)} dt + C \right).
$$

Simplify the integral:

$$
I = \int \frac{\sin(t)}{\cos(t)} e^{t} dt.
$$

Note:

$$
\frac{\sin(t)}{\cos(t)} = \tan(t),
$$

thus:

$$
I = \int e^{t} \tan(t) dt.
$$

This integral isn't straightforward but can be approached via substitution or integration by parts.

Determining the Constant Using Initial Conditions

Given $X(1.25)=4$ and $\frac{dx}{dt}|_{t=1.25} = 1$, the solution process involves:

- Calculating the indefinite integral I at $t=1.25$,
- Applying the initial condition to find C,
- Confirming that the derivative at $t=1.25$ matches 1.

This process confirms that the solution is uniquely specified by the initial conditions.

Implications and Significance

Existence and Uniqueness Confirmed

Given the continuity of the coefficients $P(t)$ and $Q(t)$ in an interval around $t=1.25$, and the initial conditions specified at this point, the differential equation admits a unique solution in this interval. This guarantees the well-posedness of the IVP, ensuring that the model describing the system's behavior is both solvable and predictable.

Relevance in Applied Contexts

- Physics: Many physical systems, such as oscillating circuits or mechanical systems with damping, are modeled through linear differential equations with variable coefficients akin to this problem.

- Engineering: Control systems often rely on the unique solutions to differential equations to predict system responses accurately.

- Mathematics: The problem provides an excellent illustration of classical existence and uniqueness theorems, emphasizing the importance

[The Ivp Sin T Dx Dt Cos T Dx Dt Sin T X Tan T X 1 25 4dx Dt 1 25 1has A Unique Solution Defined]

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