The Plane That Passes Through The Point (1, 5, 1) And Is Perpendicular To The Planes 2x + Y - 2z = 2

The Plane That Passes Through The Point (1, 5, 1) And Is Perpendicular To The Planes 2x + Y - 2z = 2 is a fascinating topic in analytical geometry, involving the determination of a specific plane based on given conditions. This problem combines the concepts of plane equations, normal vectors, and perpendicularity in three-dimensional space. Understanding how to derive such a plane requires a solid grasp of vector algebra, the geometric interpretation of planes, and the relationship between planes and their normals. In this comprehensive article, we will explore the process of finding this particular plane step-by-step, discussing relevant concepts, methods, and applications in detail.

Understanding the Basics: Planes in 3D Space

What Is a Plane?

A plane in three-dimensional space is a flat, two-dimensional surface extending infinitely in all directions. It can be mathematically represented by an equation of the form:

$$ax + by + cz + d = 0$$

where (a, b, c) is the normal vector to the plane, and d is a scalar constant.

Normal Vectors and Plane Equations

The normal vector of a plane is perpendicular to the plane's surface. Its components correspond to the coefficients a, b, c in the plane equation. The importance of the normal vector lies in its ability to determine the orientation of the plane, and it plays a crucial role when dealing with perpendicularity conditions.

Given Conditions and Objectives

Problem Restatement

Our goal is to find the equation of a plane that:
- Passes through the point $(1, 5, 1)$
- Is perpendicular to the plane $2x + y - 2z = 2$

Key Points to Consider

- The plane passes through a specific point.
- The plane must be orthogonal (perpendicular) to another plane, which influences its normal vector.

Step-by-Step Solution Approach

1. Understanding the Normal Vectors

The first step is to analyze the given plane $2x + y - 2z = 2$. Its normal vector is derived directly from its coefficients:

$$\vec{n}_1 = (2, 1, -2)$$

Since the new plane must be perpendicular to this plane, its normal vector must satisfy a specific orthogonality condition.

2. Conditions for Perpendicular Planes

Two planes are perpendicular if their normal vectors are orthogonal, meaning their dot product is zero:

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$

where $\vec{n}_2$ is the normal vector of the plane we want to find.

Therefore, the normal vector $\vec{n}_2 = (a, b, c)$ of the required plane must satisfy:

$$(2, 1, -2) \cdot (a, b, c) = 0$$

which expands to:

$$2a + b - 2c = 0$$

This is our first key equation.

3. Find the Normal Vector of the Required Plane

The plane passes through the point $(1, 5, 1)$, so its equation can be written as:

$$a(x - 1) + b(y - 5) + c(z - 1) = 0$$

or, equivalently:

$$a x + b y + c z + d = 0$$

where the constant d can be found by substituting the point:

$$a(1) + b(5) + c(1) + d = 0$$

which simplifies to:

$$a + 5b + c + d = 0$$

or

$$d = - (a + 5b + c)$$

The key now is to determine a, b, c that satisfy the orthogonality condition and then write the plane equation.

4. Determining the Normal Vector $\vec{n}_2$

From the orthogonality condition:
$$2a + b - 2c = 0$$
we can express one variable in terms of others. For example, solving for b:
$$b = -2a + 2c$$

Now, a and c can be chosen freely, and b is determined by this relation.

Parametric Form of the Normal Vector:
Let:
- $a = t$ (a free parameter)
- $c = s$ (another free parameter)
then:
$$b = -2t + 2s$$

Thus, the normal vector becomes:
$$\vec{n}_2 = (t, -2t + 2s, s)$$

Choosing specific values for simplicity:
- Set $t = 1$, $s = 0$:
$$\vec{n}_2 = (1, -2, 0)$$
- Or $t = 0$, $s = 1$:
$$\vec{n}_2 = (0, 2, 1)$$

These are just examples; any scalar multiples or linear combinations are valid as long as the orthogonality condition holds.

Constructing the Equation of the Plane

Using Normal Vector and Point

Suppose we select $\vec{n}_2 = (1, -2, 0)$. The plane passing through $(1, 5, 1)$ is:
$$1(x - 1) - 2(y - 5) + 0(z - 1) = 0$$
which simplifies to:
$$(x - 1) - 2(y - 5) = 0$$
$$x - 1 - 2y + 10 = 0$$
$$x - 2y + 9 = 0$$

This is one possible plane satisfying the conditions.

Alternatively, choosing $\vec{n}_2 = (0, 2, 1)$:

$$0(x - 1) + 2(y - 5) + 1(z - 1) = 0$$

$$0 + 2y - 10 + z - 1 = 0$$

$$2y + z - 11 = 0$$

Summary of possible planes:
- $x - 2y + 9 = 0$
- $2y + z - 11 = 0$

Any scalar multiple of these normal vectors will generate an equivalent plane.

Applications and Importance of the Derived Plane

In Geometry and Engineering

Understanding the relationship between planes, especially perpendicular planes passing through specific points, has numerous applications:
- Designing structural elements that are orthogonal for stability.
- Modeling intersections in CAD (Computer-Aided Design) software.
- Analyzing spatial relationships in robotics and navigation.

In Mathematical Analysis

The process demonstrates key concepts:
- Vector orthogonality and dot product.
- Plane equations and point-normal form.
- Parameterization of solutions for geometric problems.

Summary and Key Takeaways

- The normal vector of a plane perpendicular to another is orthogonal to that plane's normal vector.
- The general form of the plane passing through a point uses the point-normal form.
- Parameterization allows for flexible solutions, which can be tested and refined based on specific criteria.
- The approach combines algebraic and geometric reasoning to find the desired plane equation.

Conclusion

Finding a plane that passes through a specific point and is perpendicular to a given plane involves understanding the relationship between their normal vectors. By applying the dot product condition for orthogonality, expressing one normal vector in terms of free parameters, and substituting the point coordinates, we derive the equation of the required plane. This problem exemplifies fundamental principles in multivariable calculus and vector algebra, with practical relevance in fields ranging from architecture to computer graphics. Whether you are a student studying geometry or an engineer designing complex structures, mastering these methods provides a strong foundation for analyzing and constructing spatial relationships in three-dimensional space.

Keywords for SEO Optimization:
- plane passing through point
- perpendicular plane
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Frequently Asked Questions

What is the general form of the plane that passes through the point (1, 5, 1) and is perpendicular to the plane 2x + y - 2z = 2?

The plane's normal vector must be parallel to the given plane's normal vector, which is (2, 1, -2). Therefore, the plane passing through (1, 5, 1) with this normal vector has the equation: 2(x - 1) + 1(y - 5) - 2(z - 1) = 0.

How do you find the equation of the plane passing through (1, 5, 1) and perpendicular to the plane 2x + y - 2z = 2?

Since the plane is perpendicular to the given plane, its normal vector is the same as that of the given plane, which is (2, 1, -2). Using point-normal form, the equation is: 2(x - 1) + 1(y - 5) - 2(z - 1) = 0.

What is the explicit equation of the plane passing through (1, 5, 1) and perpendicular to 2x + y - 2z = 2?

Expanding the point-normal form yields: 2x + y - 2z = 2 + (21 + 15 - 21) = 2 + (2 + 5 - 2) = 2 + 5 = 7. Therefore, the plane's equation is: 2x + y - 2z = 7.

Why does the plane passing through (1, 5, 1) perpendicular to 2x + y - 2z = 2 have the normal vector (2, 1, -2)?

Because two planes are perpendicular if their normal vectors are orthogonal or aligned accordingly, and since the plane is perpendicular to the given plane, its normal vector must be parallel to the original plane's normal vector, which is (2, 1, -2).

How can I verify that the plane passing through (1, 5, 1) with normal vector (2, 1, -2) is perpendicular to the plane 2x + y - 2z = 2?

To verify, check if the normals are orthogonal by computing their dot product: (2, 1, -2) · (2, 1, -2) = 22 + 11 + (-2)(-2) = 4 + 1 + 4 = 9, which is not zero. Since the normals are identical (or parallel), the planes are perpendicular if their normals are aligned, confirming the perpendicularity.

Can the plane passing through (1, 5, 1) and perpendicular to 2x + y - 2z = 2 be written in the form ax + by + cz = d?

Yes. Since the normal vector is (2, 1, -2), and it passes through (1, 5, 1), the equation is: 2(x - 1) + 1(y - 5) - 2(z - 1) = 0, which simplifies to 2x + y - 2z = 7.

Additional Resources

The Plane That Passes Through The Point (1, 5, 1) And Is Perpendicular To The Planes 2x + Y - 2z = 2

In the fascinating realm of three-dimensional geometry, understanding the relationships between planes and points is fundamental. One intriguing problem involves determining a specific plane that passes through a given point and maintains a perpendicular orientation to a particular set of planes. Such problems are not only central to theoretical mathematics but also have practical applications in fields like engineering, computer graphics, and physics. Today, we explore the problem: "The plane that passes through the point (1, 5, 1) and is perpendicular to the plane 2x + y - 2z = 2." We will delve into the geometric principles, the step-by-step solution process, and the significance of this problem in a clear, comprehensive manner.

Understanding the Geometric Foundations

Before tackling the problem directly, it's essential to revisit some key concepts in three-dimensional geometry that form its backbone.

Planes and Their Normal Vectors

A plane in three-dimensional space can be represented by a linear equation of the form:

$$ Ax + By + Cz + D = 0 $$

Here, the coefficients A, B, and C define a vector normal (perpendicular) to the plane, known as the normal vector. The significance of the normal vector lies in its ability to describe the orientation of the plane: any plane with the same normal vector is parallel, and the direction of this vector indicates the plane's tilt relative to the coordinate axes.

Perpendicular Planes and Normal Vectors

Two planes are perpendicular if their normal vectors are orthogonal, i.e., their dot product is zero. Mathematically, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors of the two planes, then:

$$ \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 $$

This property allows us to determine relationships between planes based solely on their normal vectors, simplifying many problems in three-dimensional geometry.

The Given Plane: 2x + y - 2z = 2

The plane provided in the problem, $2x + y - 2z = 2$, has a normal vector:

$$ \mathbf{n}_1 = (2,\, 1,\, -2) $$

Understanding this normal vector is crucial, as the plane we are asked to find must be perpendicular to it. Therefore, our target plane's normal vector must be orthogonal to $\mathbf{n}_1$.

Defining the Target Plane: Geometric Constraints

The problem stipulates that the desired plane passes through a specific point, (1, 5, 1), and is perpendicular to the given plane $2x + y - 2z = 2$. To formalize this:

- The plane must pass through $P_0 = (1, 5, 1)$.
- Its normal vector, $\mathbf{n}_2$, must satisfy:

$$\mathbf{n}_2 \cdot \mathbf{n}_1 = 0$$

which ensures perpendicularity with the given plane.

Key Insight: Since the normal vector of the plane we seek must be orthogonal to $\mathbf{n}_1$, it essentially lies in the plane perpendicular to $\mathbf{n}_1$.

Therefore:

$$\mathbf{n}_2 \cdot (2, 1, -2) = 0$$

The problem reduces to finding all vectors $\mathbf{n}_2 = (A, B, C)$ satisfying this orthogonality condition.

Step-by-Step Solution Process

Let's now methodically work through the problem to find the explicit equation of the required plane.

Step 1: Find the Normal Vector $\mathbf{n}_2$

As established, $\mathbf{n}_2$ must satisfy:

$$2A + 1 \times B - 2C = 0$$

or simplified:

$$2A + B - 2C = 0$$

This is a linear relation among A, B, and C. The solutions form a plane in the space of normal vectors, providing infinitely many possible normal vectors for the plane we seek.

Step 2: Express $\mathbf{n}_2$ in terms of two parameters

To find a concrete normal vector, pick two parameters, say A and C, and express B in terms of them:

$$B = -2A + 2C$$

Thus,

$$\mathbf{n}_2 = (A,\, -2A + 2C,\, C)$$

where A and C are arbitrary real numbers.

Step 3: Write the general equation of the plane

The general form of the plane passing through $P_0 = (1, 5, 1)$ with normal vector $\mathbf{n}_2 = (A, B, C)$ is:

$$A(x - 1) + B(y - 5) + C(z - 1) = 0$$

Substituting $B = -2A + 2C$:

$$A(x - 1) + (-2A + 2C)(y - 5) + C(z - 1) = 0$$

This equation represents a family of planes parametrized by A and C. To obtain specific examples, choose particular values for A and C.

Step 4: Choosing specific parameters for clarity

For simplicity, select convenient values:

- Let $A = 1$
- Let $C = 0$

Then,

$$B = -2(1) + 2(0) = -2$$

The normal vector:

$$\mathbf{n}_2 = (1, -2, 0)$$

The plane equation:

$$1(x - 1) - 2(y - 5) + 0(z - 1) = 0$$

Simplify:

$$(x - 1) - 2(y - 5) = 0$$

$$x - 1 - 2y + 10 = 0$$

$$x - 2y + 9 = 0$$

This is one specific plane passing through (1, 5, 1) and perpendicular to the original plane.

Similarly, choosing different values for A and C would generate other valid planes with the same perpendicularity property.

Interpreting the Result

The process illustrates a fundamental geometric principle: the set of all planes passing through a point and perpendicular to a given plane form a family characterized by their normal vectors lying in a specific plane orthogonal to the original plane's normal vector.

The explicit example:

$$x - 2y + 9 = 0$$

is just one member of this family. In general, the collection of all such planes can be expressed parametrically in terms of A and C, as shown.

Practical Implications and Applications

Understanding how to determine planes with specific relationships to others is critical in various practical contexts:

- Engineering Design: Ensuring parts are aligned perpendicularly in three-dimensional assemblies.
- Computer Graphics: Calculating surface orientations and reflections.

- Physics: Modeling electromagnetic or gravitational fields where perpendicular planes represent boundary conditions.
- Robotics and Navigation: Planning paths and orientations respecting spatial constraints.

This problem exemplifies how geometric reasoning, algebraic manipulation, and parameterization come together to solve complex spatial problems efficiently.

Conclusion

Determining the plane passing through a specific point and perpendicular to a given plane demonstrates the elegant interplay between algebra and geometry in three-dimensional space. By focusing on the properties of normal vectors and their orthogonality, we can derive general solutions and particular instances, providing a robust framework for solving similar spatial problems.

The key takeaway is that the normal vector of the desired plane must be orthogonal to the normal of the given plane, and passing through the point constrains its position. This approach not only offers a clear pathway to the solution but also deepens our understanding of spatial relationships—an essential skill in both pure mathematics and applied sciences.

Whether in designing complex structures, visualizing computer-generated environments, or exploring the fundamentals of physics, mastering these principles empowers us to navigate and manipulate the three-dimensional world with confidence and precision.

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