Use A Triple Integral To Find The Volume Of The Given Solid.The Solid Enclosed By The Paraboloidsy =

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Understanding the Problem: Volume Enclosed by Paraboloids

When dealing with complex three-dimensional shapes, calculus provides powerful tools to compute their volume. One particularly effective method involves using triple integrals. This technique is especially useful when the solid is bounded by surfaces such as paraboloids, which are quadratic surfaces opening either upward or downward.

In this article, we will explore how to use a triple integral to find the volume of the solid enclosed by two paraboloids. The specific example involves the paraboloids:

- $y = x^2 + z^2$
- $y = 4 - x^2 - z^2$

Our goal is to determine the volume enclosed between these two surfaces.

Visualizing the Surfaces and the Solid

Graphing the Paraboloids

Understanding the geometric configuration of the paraboloids is crucial:

- The paraboloid $y = x^2 + z^2$ opens upward, with its vertex at the origin $(0,0,0)$.
- The paraboloid $y = 4 - x^2 - z^2$ opens downward, with its vertex at $(0,4,0)$.

When these surfaces are plotted, they intersect in a closed, symmetric shape that resembles a "lens" or "capsule." The intersection occurs where the two surfaces meet, i.e., where:

$$
x^2 + z^2 = 4 - x^2 - z^2
$$

which simplifies to:

$$
2(x^2 + z^2) = 4
$$

or

$$
x^2 + z^2 = 2
$$

This circle in the xz-plane indicates the boundary of the enclosed solid.

Identifying the Region of Integration

The intersection circle:

$$
x^2 + z^2 = 2
$$

serves as the outer boundary in the xz-plane. The y-coordinate varies between the lower paraboloid $y = x^2 + z^2$ and the upper paraboloid $y = 4 - x^2 - z^2$.

Setting Up the Triple Integral

Coordinate System Choice

Because of the circular symmetry in x and z, cylindrical coordinates are most suitable for this problem:

$$
\begin{cases}
x = r \cos \theta \\
z = r \sin \theta \\
$$

y = y
\end{cases}
\]

where:

- \(r \geq 0 \)
- \(0 \leq \theta \leq 2\pi \)
- \(y \) varies between the two paraboloids

In cylindrical coordinates:

- \(x^2 + z^2 = r^2 \)

The paraboloids become:

\[
\text{Lower surface: } y = r^2
\]
\[
\text{Upper surface: } y = 4 - r^2
\]

The intersection circle in the \(xy \)-plane is at \(r = \sqrt{2} \).

Determining Limits of Integration

- Radial coordinate \(r \): from 0 to \(\sqrt{2} \).
- Angular coordinate \(\theta \): from 0 to \(2\pi \).
- Vertical coordinate \(y \): from the lower paraboloid \(y = r^2 \) up to the upper paraboloid \(y = 4 - r^2 \).

Constructing the Triple Integral

The volume \(V \) of the solid is given by:

\[
V = \iiint_{D} dV
\]

In cylindrical coordinates, the differential volume element is:

\[

$$
dV = r \, dy \, dr \, d\theta
$$

Therefore, the triple integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \int_{r^2}^{4 - r^2} r \, dy \, dr \, d\theta
$$

Evaluating the Integral Step-by-Step

Integrate with respect to y

$$
\int_{y = r^2}^{4 - r^2} r \, dy = r \left[y \right]_{r^2}^{4 - r^2} = r \left((4 - r^2) - r^2 \right) = r (4 - 2r^2)
$$

Integrate with respect to r

$$
V = \int_{0}^{2\pi} \left(\int_{0}^{\sqrt{2}} r (4 - 2r^2) dr \right) d\theta
$$

Calculate the inner integral:

$$
\int_{0}^{\sqrt{2}} r (4 - 2r^2) dr
$$

Distribute:

$$
\int_{0}^{\sqrt{2}} (4r - 2r^3) dr
$$

Integrate term-by-term:

$$
\left[2r^2 - \frac{1}{2} r^4 \right]_0^{\sqrt{2}}
$$

Evaluate at the bounds:

$$
2 (\sqrt{2})^2 - \frac{1}{2} (\sqrt{2})^4 = 2 \times 2 - \frac{1}{2} \times (2)^2 = 4 - \frac{1}{2} \times 4 = 4 - 2 = 2
$$

Integrate with respect to θ

Since the integrand is independent of θ:

$$
V = \int_{0}^{2\pi} 2 \, d\theta = 2 \times 2\pi = 4\pi
$$

Final Result and Interpretation

The volume enclosed between the paraboloids $y = x^2 + z^2$ and $y = 4 - x^2 - z^2$ is:

$$
\boxed{V = 4\pi}
$$

This result confirms the symmetry and the elegance of calculus in solving three-dimensional volume problems with quadratic surfaces.

Conclusion and Applications

Using triple integrals to find volumes enclosed by paraboloids demonstrates the power of multivariable calculus. This technique can be extended to various other surfaces and shapes, including ellipsoids, hyperboloids, and more complex bounded regions.

Applications include:

- Engineering: calculating the volume of objects with curved surfaces
- Physics: determining mass or charge distributions within bounded regions
- Geometry: understanding properties of quadratic surfaces
- Computer Graphics: volume rendering and modeling

Key takeaways:

- Recognize the symmetry of the problem to choose an appropriate coordinate system
- Find the intersection boundary to determine limits
- Set up the integral carefully, considering the order of integration
- Evaluate step-by-step for clarity and accuracy

Additional Tips for Solving Similar Problems

- Always visualize the surfaces and their intersections.
- Use symmetry to reduce computational complexity.
- Convert to cylindrical or spherical coordinates when dealing with circular or spherical symmetries.
- Carefully determine the limits for each variable based on the surface equations.
- Remember the Jacobian determinant when changing coordinate systems; in cylindrical coordinates, $dV = r\,dy\,dr\,d\theta$.

By mastering the use of triple integrals in these contexts, students and professionals can confidently analyze and compute volumes of complex solids bounded by quadratic surfaces.

Frequently Asked Questions

How do you set up a triple integral to find the volume enclosed by the paraboloids $y = x^2 + z^2$ and $y = 4 - x^2 - z^2$?

First, find the intersection curve of the paraboloids to determine the bounds. Convert to cylindrical coordinates if convenient. The volume can be found by integrating over the region where y varies between $x^2 + z^2$ and $4 - x^2 - z^2$, with x and z covering the intersection area. The triple integral setup typically involves integrating y from the lower paraboloid to the upper one, and x and z over the projection of the intersection circle.

What coordinate system is most suitable for solving the volume between paraboloids $y = x^2 + z^2$ and $y = 4$

- x^2 - z^2?

Cylindrical coordinates are most suitable because the paraboloids are symmetric around the z-axis, simplifying the equations and bounds of integration.

How do you find the limits of integration for y when using a triple integral to find the volume between the paraboloids?

The limits of y are from the lower paraboloid $y = x^2 + z^2$ to the upper paraboloid $y = 4 - x^2 - z^2$. These bounds depend on the projection in the x-z plane, typically a circle where the paraboloids intersect.

How do you determine the intersection points of the paraboloids $y = x^2 + z^2$ and $y = 4 - x^2 - z^2$?

Set the two equations equal: $x^2 + z^2 = 4 - x^2 - z^2$. Solving gives $2(x^2 + z^2) = 4$, so $x^2 + z^2 = 2$. The intersection occurs on the circle $x^2 + z^2 = 2$ at $y = 2$.

What are the steps involved in computing the volume enclosed by the paraboloids using a triple integral?

1. Find the intersection curve to determine bounds. 2. Convert to appropriate coordinates (cylindrical). 3. Set up the triple integral with proper limits for r, θ, and y. 4. Integrate step-by-step, starting from the innermost integral outward to find the volume.

How does symmetry help in evaluating the triple integral for the volume of the solid between the paraboloids?

Symmetry around the z-axis allows simplifying the bounds and integrand, often reducing the problem to integrating over a circular region in the x-z plane, which simplifies the limits and potentially the integrand.

What is the general form of the triple integral to find the volume between the paraboloids $y = x^2 + z^2$ and $y = 4 - x^2 - z^2$?

The volume $V = \iiint_D dV$, where D is the region between the paraboloids. In cylindrical coordinates, this becomes $V = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} \int_{y=r^2}^{4 - r^2} r \, dy \, dr \, d\theta$.

How can you verify that your limits of integration correctly capture the volume between the paraboloids?

Check the intersection points to ensure the bounds in r and y correspond correctly. Plot or analyze the projection in the x-z plane to confirm the region is fully covered. Also, ensure that the limits for y are from the lower to the upper paraboloid over that region.

Why is converting to cylindrical coordinates advantageous in this problem?

Because the paraboloids are symmetric around the z-axis, cylindrical coordinates simplify the equations and limits, making the integrals easier to evaluate compared to Cartesian coordinates.

Can the volume between paraboloids be found using a double integral instead of a triple integral?

Yes, by integrating the cross-sectional area of the region in the x-z plane (a double integral) and multiplying by the height between the paraboloids at each point, effectively reducing the problem to a double integral over the projection region.

Additional Resources

Use A Triple Integral To Find The Volume Of The Given Solid: An In-Depth Investigation

In the realm of multivariable calculus, the problem of determining the volume of a solid enclosed by complex surfaces is both fundamental and challenging. One particularly elegant method involves the use of triple integrals, which allow for precise volume calculations of three-dimensional regions bounded by surfaces such as paraboloids. This article explores the process of employing triple integrals to find the volume of a solid enclosed by paraboloids, with a focus on the specific case of the surface defined by \(y = \) (the exact paraboloid equation). Although the problem as stated is incomplete, the methodology remains universally applicable and adaptable to various paraboloidal configurations.

Understanding the Geometric Context:

Paraboloids and Their Enclosed Regions

What Are Paraboloids?

A paraboloid is a quadric surface that can be represented by a quadratic equation in three variables. The most common forms are:

- Elliptic paraboloid: $z = ax^2 + by^2$
- Hyperbolic paraboloid: $z = ax^2 - by^2$

In the context of volume calculation, the focus is often on elliptic paraboloids that open upward or downward, creating bounded regions when two such surfaces intersect.

The Enclosed Solid: General Characteristics

When two paraboloids intersect, the region enclosed between them can be visualized as a "lens-shaped" solid. To compute its volume via triple integrals, it is crucial to:

- Identify the equations of the bounding paraboloids.
- Determine their intersection curve.
- Choose an appropriate coordinate system to simplify the integral.

In many cases, cylindrical or spherical coordinates are advantageous, especially if the paraboloids are symmetric around an axis.

Setting Up the Integral: From Geometry to Calculus

Step 1: Defining the Surfaces

Suppose the problem involves two paraboloids:

1. $z = y = x^2 + y^2$ (a paraboloid opening upward)
2. $z = y = c$ (a horizontal plane or paraboloid opening downward)

Alternatively, for a more specific case, consider:

- $z = x^2 + y^2$ (a standard upward opening paraboloid)
- $z = 2 - x^2 - y^2$ (a downward opening paraboloid)

These two surfaces intersect where:

$$x^2 + y^2 = 2 - x^2 - y^2 \Rightarrow 2x^2 + 2y^2 = 2 \Rightarrow x^2 + y^2 = 1$$

This circle $x^2 + y^2 = 1$ defines the boundary in the xy-plane for the enclosed region.

Step 2: Choosing the Coordinate System

Given the symmetry around the z-axis, cylindrical coordinates are typically the most natural choice:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

with $r \geq 0$, $0 \leq \theta \leq 2\pi$, and z bounded between the two paraboloids.

Step 3: Expressing Surfaces in Cylindrical Coordinates

For the example:

- $z = r^2$
- $z = 2 - r^2$

The intersection occurs where:

$$r^2 = 2 - r^2 \Rightarrow 2r^2 = 2 \Rightarrow r^2 = 1 \Rightarrow r = 1$$

Thus, the projection of the solid onto the xy-plane is the disk $0 \leq r \leq 1$.

Step 4: Determining the Limits of Integration

- Radial coordinate r: from 0 to 1
- Angular coordinate θ: from 0 to 2π
- Vertical coordinate z: from the lower paraboloid $z = r^2$ to the

upper paraboloid $z = 2 - r^2$

Constructing the Triple Integral

Integral Expression

The volume V is given by:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} \int_{z = r^2}^{z = 2 - r^2} r \, dz \, dr \, d\theta
$$

where:

- r is the Jacobian determinant for cylindrical coordinates.
- The limits for z are from the lower paraboloid to the upper paraboloid.

Step-by-Step Calculation

1. Integrate with respect to z:

$$
\int_{z = r^2}^{z = 2 - r^2} r \, dz = r \left[z \right]_{r^2}^{2 - r^2} = r \left((2 - r^2) - r^2 \right) = r (2 - 2r^2) = 2r - 2 r^3
$$

2. Integrate with respect to r:

$$
\int_{0}^{1} (2r - 2 r^3) dr = \left[r^2 - \frac{1}{2} r^4 \right]_0^1 = (1 - \frac{1}{2}) - 0 = \frac{1}{2}
$$

3. Integrate with respect to θ:

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

4. Final volume:

$$
V = 2\pi \times \frac{1}{2} = \pi
$$

Thus, the volume enclosed by the two paraboloids is π.

Extensions and Generalizations

Handling More Complex Paraboloids

The methodology described can be extended to different paraboloid configurations by:

- Adjusting the equations of the surfaces.
- Computing the intersection boundary in the xy-plane.
- Choosing the most suitable coordinate system based on symmetry.

Alternative Coordinate Systems

While cylindrical coordinates are most effective for rotationally symmetric paraboloids, other systems may be preferable:

- Spherical coordinates: For paraboloids symmetric about a point.
- Elliptic cylindrical coordinates: When surfaces are elongated along one axis.

Numerical Integration

In cases where the surfaces are too complicated for analytical solutions, numerical methods such as Monte Carlo integration or adaptive quadrature can approximate the volume accurately.

Conclusion: The Power of Triple Integrals in Surface Enclosure Problems

Using triple integrals to find the volume of a solid enclosed by paraboloids exemplifies the profound synergy between geometric intuition and calculus. This approach not only offers exact solutions in symmetric cases but also provides a framework adaptable to more complex geometries.

By carefully identifying the intersection curves, choosing optimal coordinate systems, and setting precise limits, mathematicians and engineers can solve a broad class of volume problems with confidence. The case study of paraboloids demonstrates the elegance and efficacy of triple integrals as a fundamental tool in multivariable calculus.

Whether applied in theoretical research, engineering design, or physical modeling, the technique of integrating over three-dimensional regions enclosed by surfaces remains a cornerstone of mathematical analysis.

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Author's Note: This detailed exploration underscores the importance of understanding the geometric nature of the problem, selecting suitable coordinate systems, and methodically constructing the integral bounds. Mastery of these steps enables precise calculation of volumes in complex three-dimensional environments.

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