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In the realm of mathematical analysis, power series expansions serve as a powerful tool for approximating functions near specific points. The process of expanding a function into its power series form allows us to analyze its behavior, compute limits, derivatives, integrals, and facilitate numerical approximations. The focus here is on employing a given equation—referred to as Eq. (1) from the text—to expand a function into a power series centered at C = 0 (the origin), and subsequently, to determine the nature and coefficients of this series. This article aims to elucidate the methodology step-by-step, providing a comprehensive understanding of the process, including the derivation of the coefficients and the convergence considerations involved.

Understanding the Context and Purpose of Power Series Expansion

What Is a Power Series?

A power series is an infinite sum of the form:

```
f(x) = Σ (a_n)(x - c)^n, where n = 0 to ∞
```

Here, c is the center of the expansion, and a_n are the coefficients that determine the series' terms. When $c = 0$, the series simplifies to:

```
f(x) = Σ a_n x^n
```

which is called a Maclaurin series.

Why Expand Around C = 0?

Centering the power series at zero often simplifies calculations, especially when the function's derivatives are more manageable at zero. Many functions have known Maclaurin series expansions, or their derivatives are easier to compute at zero, making this approach advantageous for approximations and theoretical analysis.

Equation (1) From the Text: The Foundation of the Expansion

Interpreting Eq. (1)

While the original text's Eq. (1) is not explicitly provided here, it typically represents a fundamental relation for function expansion, often a form of the Taylor series or a generating function. For illustrative purposes, assume Eq. (1) is the Taylor series expansion formula centered at C, which is:

$$f(x) = \sum_{n=0}^{\infty} (f^{(n)}(c)/n!)(x - c)^n$$

This formula states that any sufficiently smooth function can be expressed as an infinite sum involving its derivatives at the point c.

Adapting Eq. (1) for Center C = 0

To expand the function into a power series centered at zero, set c = 0 in the general Taylor series:

$$f(x) = \sum_{n=0}^{\infty} (f^{(n)}(0)/n!)\, x^n$$

This is the Maclaurin series expansion, which is particularly useful for functions that are analytic at zero.

Step-by-Step Procedure for Expanding the Function

1. Identify the Function and Its Derivatives

- Determine the function $f(x)$ to be expanded.
- Calculate the derivatives $f^{(n)}(0)$ for n = 0, 1, 2, ...
- Evaluate these derivatives at zero to find the coefficients.

2. Compute the Derivatives at Zero

For many functions, derivatives at zero can be computed directly or via known derivative rules. Examples include:

- **Exponential functions:** $f(x) = e^x$

- **Sine and cosine:** $f(x) = \sin x$, $f(x) = \cos x$
- **Rational functions, algebraic functions, etc.**

3. Apply the Maclaurin Series Formula

- Plug the derivatives into the series formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

- Express the series explicitly with the coefficients determined.

4. Determine the Radius of Convergence

Use ratio or root tests to find the interval within which the series converges to the function.

Examples of Power Series Expansion

Example 1: Exponential Function

Let $f(x) = e^x$.

- Derivatives: $f^{(n)}(x) = e^x$
- At zero: $f^{(n)}(0) = 1$
- Series expansion:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

This power series converges for all real x, with an infinite radius of convergence.

Example 2: Sine Function

Let $f(x) = \sin x$.

- Derivatives at zero follow a pattern:
  - $f^{(0)}(0) = 0$
  - $f^{(1)}(0) = 1$
  - $f^{(2)}(0) = 0$
  - $f^{(3)}(0) = -1$
  - and so on, following the cycle of derivatives.
- Series expansion:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Determining the Coefficients: Practical Approaches

Method 1: Derivative Calculation

The most straightforward way involves computing derivatives at zero directly. For functions with simple derivatives, this approach is efficient and reliable.

Method 2: Recognizing Standard Series

Many common functions have well-known power series expansions. Recognizing these can save computational effort and provide immediate coefficients.

Method 3: Using Generating Functions or Known Series

- Leverage existing generating functions or series representations to derive the coefficients.
- This approach is especially useful for special functions like Bessel functions, Legendre polynomials, etc.

Convergence and Validity of the Power Series Expansion

Radius of Convergence

The power series converges within a certain radius centered at zero, determined by the distance to the nearest singularity of the function in the complex plane. Techniques such as the ratio test or root test are used to find this radius.

Analyticity and Function Behavior

- A function must be analytic at zero for a Maclaurin series to converge to the function in some neighborhood.
- If the function has singularities close to zero, the series may only converge within a limited radius.

Examples of Convergence Domains

- *e^x*: entire function, converges everywhere.
- *1/(1 - x)*: converges for $|x| < 1$.
- *ln(1 + x)*: converges for $|x| < 1$.

Advanced Topics and Applications

Using Power Series for Approximation

Power series expansions allow for polynomial approximations of functions, which are highly useful in numerical analysis and computational methods.

Analytic Continuation and Series Extension

Series can sometimes be extended beyond their initial radius of convergence through techniques such as analytic continuation.

Series in Differential Equations

Power series methods are fundamental in solving differential equations where solutions are expressed as series expansions around specific points.

Conclusion

Applying Eq. (1) from the text to expand a function into a power series centered at $C = 0$ involves leveraging the Taylor (or Maclaurin) series formula, calculating derivatives at zero, and expressing the function as an infinite sum of powers of x. This process provides insights into the function's behavior, simplifies complex functions into manageable polynomials, and underpins many theoretical and practical applications in mathematics, physics, and engineering. By carefully determining the derivatives and understanding the convergence domain, one can effectively utilize power series expansions to analyze and approximate functions with precision and confidence.