Use The Method Of Undetermined Coefficients To Find The Solution Of The Differential Equation: Y" 4y

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When faced with a second-order linear differential equation with constant coefficients, such as Y" + 4y = 0, the method of undetermined coefficients provides a powerful technique for finding its general solution. This method is especially effective for equations with non-homogeneous terms, but it can also be adapted for homogeneous equations by understanding their characteristic equations. In this comprehensive guide, we will explore how to apply the method of undetermined coefficients step-by-step, understand its theoretical basis, and see practical examples to solidify your understanding. Whether you're a student preparing for exams or a professional working on differential equations, mastering this technique is essential for solving a wide range of differential equations efficiently.

Understanding the Differential Equation Y" + 4y = 0

This differential equation is a classic example of a homogeneous second-order linear differential equation with constant coefficients. Its structure makes it suitable for solution via the characteristic equation method, but understanding the method of undetermined coefficients is equally important, especially for non-homogeneous variations.

Key Concepts

- Linear Differential Equations: Equations where the unknown function and its derivatives appear linearly.
- Homogeneous vs. Non-Homogeneous: Homogeneous equations have zero on the right side; non-homogeneous have a non-zero function.
- Characteristic Equation: An algebraic equation derived from the differential equation that helps find the general solution.

For the equation Y" + 4y = 0, the characteristic equation is:

\[r^2 + 4 = 0 \]

which has roots $r = \pm 2i$, leading to the general solution:

$$y(t) = C_1 \cos 2t + C_2 \sin 2t$$

However, to fully understand the method of undetermined coefficients, especially for non-homogeneous equations, we need to delve deeper.

Method of Undetermined Coefficients: An Overview

The method of undetermined coefficients is a technique used to find particular solutions to non-homogeneous linear differential equations with constant coefficients, where the non-homogeneous term (right side of the equation) is of a specific type.

When to Use the Method

Use this method when the non-homogeneous term ($f(t)$) is of one of the following forms:

- Polynomial functions (e.g., $P(t)$)
- Exponential functions (e.g., e^{at})
- Sine or cosine functions (e.g., $\sin bt$, $\cos bt$)
- Combinations of these (e.g., $e^{at} \sin bt$, t^n)

Important: The method is not suitable for all functions; for more complex or non-standard functions, variation of parameters may be more appropriate.

Basic Approach

1. Solve the homogeneous equation to find the complementary solution y_c.
2. Identify the form of the non-homogeneous term $f(t)$.
3. Guess the particular solution y_p based on the form of $f(t)$, introducing undetermined coefficients.
4. Substitute y_p into the original differential equation to solve for the coefficients.
5. Write the general solution as $y = y_c + y_p$.

Applying the Method to the Equation Y" + 4y = 0

Since the equation is homogeneous, the method of undetermined coefficients is traditionally not necessary—the characteristic equation directly yields the general solution. However, for educational purposes, we can explore how this method applies if we consider a non-homogeneous version of this differential equation, such as:

$$ Y'' + 4Y = G(t) $$

where $G(t)$ is a non-homogeneous term, for example, $G(t) = \cos 2t$.

Step 1: Find the Complementary Solution y_c

Solve the homogeneous equation:

$$ Y'' + 4Y = 0 $$

Characteristic equation:

$$ r^2 + 4 = 0 \Rightarrow r = \pm 2i $$

Solution:

$$ y_c(t) = C_1 \cos 2t + C_2 \sin 2t $$

Step 2: Identify the Non-Homogeneous Term $G(t)$

Suppose:

$$ G(t) = \cos 2t $$

This is a key step, as the form of $G(t)$ influences the choice of the particular solution.

Step 3: Guess the Particular Solution y_p

Since $G(t) = \cos 2t$, and $\cos 2t$ is a solution to the homogeneous equation (as part of the complementary solution), we need to modify our guess to avoid duplication:

- Standard guess: $y_p(t) = A \cos 2t + B \sin 2t$
- Because these are solutions to the homogeneous equation, we multiply by t:

$$y_p(t) = t(A \cos 2t + B \sin 2t)$$

This ensures linear independence and allows solving for A and B.

Step 4: Compute Derivatives and Substitute

Calculate y_p' and y_p'', then substitute into the original differential equation:

$$y'' + 4y = G(t)$$

This step involves differentiating complex expressions, but the process follows standard calculus rules.

Step 5: Solve for the Undetermined Coefficients

After substitution, equate coefficients of like terms to find the values of A and B.

General Solution and Interpretation

The overall solution combines the complementary and particular solutions:

$$y(t) = y_c(t) + y_p(t)$$

where:

- $y_c(t) = C_1 \cos 2t + C_2 \sin 2t$
- $y_p(t)$ depends on the specific form of $G(t)$ and is found via the undetermined coefficients method.

Practical Examples of Using the Method of Undetermined Coefficients

Let's explore some illustrative examples to clarify the process.

Example 1: Solve $Y'' + 4Y = \cos 2t$

Step 1: Homogeneous solution:

$$y_c(t) = C_1 \cos 2t + C_2 \sin 2t$$

Step 2: Non-homogeneous term:

$$G(t) = \cos 2t$$

Step 3: Since $\cos 2t$ is part of the homogeneous solution, guess:

$$y_p(t) = t(A \cos 2t + B \sin 2t)$$

Step 4: Differentiate and substitute into the original equation to solve for A and B.

Result: After calculations, you find the particular solution, and the general solution becomes:

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + t \left(\text{coefficients} \right)$$

Example 2: Solve $Y'' + 4Y = e^{2t}$

Step 1: Homogeneous solution:

$$y_c(t) = C_1 \cos 2t + C_2 \sin 2t$$

Step 2: Non-homogeneous term:

$$G(t) = e^{2t}$$

Step 3: Guess:

$$y_p(t) = A e^{2t}$$

since e^{2t} is not part of the homogeneous solution, no modification is necessary.

Step 4: Substitute y_p into the original differential equation and solve for A.

Result: The particular solution:

$$y_p(t) = \frac{e^{2t}}{(4 - 4)}$$

which indicates a resonance, and special techniques may be needed to resolve this.

Summary and Key Takeaways

- The method of undetermined coefficients is a systematic way to find particular solutions of non-homogeneous linear differential equations with constant coefficients.
- It involves guessing a form for the particular solution based on the non-homogeneous term and solving for unknown coefficients.
- The method is most effective when the non-homogeneous term is a polynomial, exponential, sine, cosine, or their combinations.
- When the guessed form overlaps with the homogeneous solution, multiply by t to ensure linear independence.
- The overall solution is the sum of the complementary (homogeneous) and particular solutions.

Benefits of Using the Method of Undetermined Coefficients

- Efficiency: Quick and straightforward for suitable cases.
- Clarity: Clear step-by-step process that students can follow.
- Applicability: Useful in engineering, physics, and applied mathematics contexts.

Final Thoughts