Use The Method Of Undetermined Coefficients To Solve The Initial Value Problem Below. Y'' + 7y=7 Sin

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The method of undetermined coefficients is a powerful technique used to find particular solutions to linear differential equations with constant coefficients, especially when the nonhomogeneous term (forcing function) is a standard type such as polynomials, exponentials, sines, and cosines. In this article, we will explore how to employ this method to solve a specific initial value problem (IVP) of the form *Y'' + 7y = 7 Sin*. Although the problem statement appears incomplete, likely, the forcing function involves a sinusoidal term, probably *7 Sin t*. We will interpret it as such and proceed accordingly. Our goal is to find the general solution to the differential equation and then determine the particular solution using undetermined coefficients, followed by applying initial conditions to find specific constants.

Understanding the Differential Equation

Form of the Differential Equation

The differential equation in question is:

```
Y'' + 7y = 7 Sin t
```

This is a second-order linear nonhomogeneous differential equation with constant coefficients. The homogeneous part is:

```
Y'' + 7y = 0
```

and the nonhomogeneous term (forcing term) is:

```
7 Sin t
```

Characteristics of the Equation

- The characteristic equation for the homogeneous part is:

```
r^2 + 7 = 0
```

- Solutions to the characteristic equation are:

```
r = ± i√7
```

- Homogeneous solution:

```
y_h(t) = C₁ cos(√7 t) + C₂ sin(√7 t)
```

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Finding the General Solution

Step 1: Homogeneous Solution

As shown above, the homogeneous solution is:

```
y_h(t) = C₁ cos(√7 t) + C₂ sin(√7 t)
```

Step 2: Particular Solution using Undetermined Coefficients

The nonhomogeneous term is *7 Sin t*. Since the forcing function is a sine function with argument t, our particular solution will be assumed to have the form:

```
y_p(t) = A cos t + B sin t
```

where A and B are undetermined coefficients to be found.

Step 3: Derivatives of the Particular Solution

Calculate the first and second derivatives:

```
y_p'(t) = -A sin t + B cos t
```

```
y_p''(t) = -A cos t - B sin t
```

Step 4: Substitute into the Differential Equation

Substitute *y_p* and its derivatives into the original differential equation:

```
(-A cos t - B sin t) + 7(A cos t + B sin t) = 7 Sin t
```

Simplify the left side:

```
(-A cos t - B sin t) + 7A cos t + 7B sin t = 7 Sin t
```

Combine like terms:

```
(-A + 7A) cos t + (-B + 7B) sin t = 7 Sin t
```

```
(6A) cos t + (6B) sin t = 7 Sin t
```

Step 5: Equate Coefficients

Matching coefficients from both sides gives the system:

- For cos t:

```
6A = 0  ⇒  A = 0
```

- For sin t:

```
6B = 7  ⇒  B = 7/6
```

Step 6: Write the Particular Solution

Thus, the particular solution is:

```
y_p(t) = 0  cos t + (7/6) sin t = (7/6) sin t
```

Complete General Solution

The general solution to the differential equation is the sum of the homogeneous and particular solutions:

$$y(t) = y_h(t) + y_p(t) = C_1 \cos(\sqrt{7}\, t) + C_2 \sin(\sqrt{7}\, t) + (7/6) \sin t$$

Applying Initial Conditions

Step 1: Determine Initial Conditions

To find specific values for C_1 and C_2, initial conditions such as $y(0)$ and $y'(0)$ are needed. Suppose the problem provides:

- $y(0) = y_0$
- $y'(0) = y'_0$

Step 2: Evaluate the Solution at t = 0

$$y(0) = C_1 \cos(0) + C_2 \sin(0) + (7/6) \sin 0 = C_1 \;\; 1 + C_2 \;\; 0 + 0 = C_1$$

Therefore:

$$C_1 = y_0$$

Step 3: Find y'(t) and Evaluate at t = 0

$$y'(t) = -C_1 \sqrt{7} \sin(\sqrt{7}\, t) + C_2 \sqrt{7} \cos(\sqrt{7}\, t) + (7/6) \cos t$$

At t = 0:

$$y'(0) = -C_1 \sqrt{7} \;\; 0 + C_2 \sqrt{7} \;\; 1 + (7/6) \;\; 1 = C_2 \sqrt{7} + (7/6)$$

Thus:

$$C_2 \sqrt{7} + (7/6) = y'_0$$

and solving for C_2:

$$C_2 = (y'_0 - (7/6)) / \sqrt{7}$$

Summary and Final Remarks

Using the method of undetermined coefficients, we've systematically derived the particular solution to the differential equation and combined it with the homogeneous solution to obtain the general solution. The key steps involved assuming a form for the particular solution based on the forcing function, calculating derivatives, substituting into the original equation, and matching coefficients. Initial conditions allow us to resolve the arbitrary constants, yielding a specific solution tailored to the problem at hand.

This approach is versatile and can be extended to other types of forcing functions, with modifications in the assumed particular solution form as needed. The method of undetermined coefficients is especially efficient when the nonhomogeneous term is a linear combination of exponentials, polynomials, sines, and cosines, owing to the predictable structure of the particular solutions.

Additional Considerations

Handling Resonance and Repeated Roots

If the forcing function's form matches a solution to the homogeneous equation (e.g., if the forcing term is a sine or cosine with a frequency equal to that of the homogeneous solution), resonance occurs. In such cases, the assumed particular solution must be multiplied by t to account for the repeated roots, ensuring linear independence.

Limitations of the Method

- It applies primarily to equations with constant coefficients and standard forcing functions.
- Complex forcing functions might require variation of parameters instead.

Conclusion

The method of undetermined coefficients provides a straightforward process to find particular solutions for linear differential equations with constant coefficients and sinusoidal or polynomial forcing functions. By carefully choosing the form of the particular solution, computing derivatives, and matching coefficients, one can efficiently solve initial value problems like the one considered here. Mastery of this method enhances one's ability to analyze and solve a wide range of differential equations encountered in physics, engineering, and applied mathematics.

Frequently Asked Questions

What is the general form of the particular solution when using the method of undetermined coefficients for the differential equation y'' + 7y = 7 sin x?

The particular solution is assumed to be of the form $y_p = A \cos x + B \sin x$, since the right-hand side is 7 sin x and the nonhomogeneous term suggests sinusoidal solutions.

How do you determine the form of the trial solution when the right-hand side involves sine or cosine functions?

When the RHS involves sine or cosine functions, the trial particular solution is typically chosen as a linear combination of sine and cosine terms with undetermined coefficients, such as $y_p = A \cos x + B \sin x$.

What is the complementary (homogeneous) solution for the differential equation y'' + 7y = 0?

The homogeneous equation $y'' + 7y = 0$ has characteristic equation $r^2 + 7 = 0$, with roots $r = \pm i\sqrt{7}$, so the complementary solution is $y_c = C_1 \cos(\sqrt{7}\, x) + C_2 \sin(\sqrt{7}\, x)$.

How do you find the coefficients A and B in the particular solution for y'' + 7y = 7 sin x?

Substitute $y_p = A \cos x + B \sin x$ into the left side of the differential equation, compute $y_p'' + 7 y_p$, set it equal to 7 sin x, and solve for A and B by equating coefficients.

What steps are involved in solving the initial value problem using the method of undetermined coefficients?

First, find the homogeneous solution; second, assume a particular solution with undetermined coefficients; third, substitute into the differential equation to determine these coefficients; fourth, combine solutions; fifth, use initial conditions to solve for constants.

Can the method of undetermined coefficients be used if the right-hand side is a polynomial or exponential function?

Yes, but the assumed particular solution form differs: for polynomials, use polynomial guesses; for exponentials, include exponential terms; the method adapts based on the form of the nonhomogeneous term.

Additional Resources

Method of Undetermined Coefficients is a powerful technique in differential equations, especially suited for solving linear nonhomogeneous ordinary differential equations with constant coefficients. When faced with an initial value problem (IVP) such as $y'' + 7y = 7 \sin x$, this method provides a systematic approach to find particular solutions, complementing the general solution of the associated homogeneous equation. In this article, we will delve into the process of applying the method of undetermined coefficients to this specific IVP, exploring each step with clarity and analytical depth.

Understanding the Problem: The Differential Equation and Initial Conditions

Before diving into the solution process, it's essential to understand the structure of the problem at hand.

The Differential Equation

The given differential equation is:

$$
y'' + 7y = 7 \sin x
$$

This is a second-order linear differential equation with constant coefficients. The left side represents a homogeneous differential operator, while the right side, $7 \sin x$, introduces a nonhomogeneous term (forcing function).

Initial Conditions

Although not explicitly provided in the statement, initial conditions are typically necessary to determine the particular solution that satisfies the initial state of the system. For illustration purposes, suppose the initial conditions are:

$$
y(0) = y_0, \quad y'(0) = y'_0
$$

\]

The focus of the article, however, remains on solving the differential equation itself using the method of undetermined coefficients.

Decomposing the Solution: Homogeneous and Particular Solutions

Any solution to a nonhomogeneous differential equation can be expressed as the sum of:

1. Homogeneous solution (y_h): The general solution to the associated homogeneous equation $y'' + 7y = 0$.

2. Particular solution (y_p): A specific solution that accounts for the nonhomogeneous term $7 \sin x$.

Homogeneous Solution

The homogeneous equation:

$$
y'' + 7 y = 0
$$

has characteristic equation:

$$
r^2 + 7 = 0
$$

which yields roots:

$$
r = \pm i \sqrt{7}
$$

Therefore, the homogeneous solution:

$$
y_h(x) = C_1 \cos (\sqrt{7} x) + C_2 \sin (\sqrt{7} x)
$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Applying the Method of Undetermined Coefficients

Now, the core of this article focuses on finding a particular solution y_p to the nonhomogeneous equation:

$$
y'' + 7 y = 7 \sin x
$$

using the method of undetermined coefficients.

Concept Overview

The method involves guessing a form for y_p based on the form of the nonhomogeneous term $7 \sin x$. Since the forcing function involves a sine function, the particular solution is typically guessed as a linear combination of sine and cosine functions with unknown coefficients.

Step 1: Guess the Form of y_p

Given $7 \sin x$, the standard guess is:

$$
y_p(x) = A \sin x + B \cos x
$$

where A and B are undetermined coefficients to be found.

Step 2: Compute Derivatives of y_p

Calculate the first and second derivatives:

$$
y_p' = A \cos x - B \sin x
$$

$$
y_p'' = -A \sin x - B \cos x
$$

Step 3: Substitute into Differential Equation

Plugging y_p, y_p', and y_p'' into the original differential equation:

$$
(-A \sin x - B \cos x) + 7 (A \sin x + B \cos x) = 7 \sin x
$$

Simplify:

\[
-A \sin x - B \cos x + 7A \sin x + 7B \cos x = 7 \sin x
\]

Group like terms:

\[
(-A + 7A) \sin x + (-B + 7B) \cos x = 7 \sin x
\]

which simplifies to:

\[
(6A) \sin x + (6B) \cos x = 7 \sin x
\]

Step 4: Equate Coefficients

Matching the coefficients of \(\sin x \) and \(\cos x \):

\[
6A = 7 \quad \Rightarrow \quad A = \frac{7}{6}
\]

\[
6B = 0 \quad \Rightarrow \quad B = 0
\]

Thus, the particular solution is:

\[
\boxed{
y_p(x) = \frac{7}{6} \sin x
}
\]

Complete Solution and Initial Conditions

The general solution to the original IVP combines the homogeneous and particular solutions:

\[
y(x) = y_h(x) + y_p(x) = C_1 \cos (\sqrt{7} x) + C_2 \sin (\sqrt{7} x) + \frac{7}{6} \sin x
\]

To fully determine C_1 and C_2, initial conditions are applied. For example, if $y(0)$ and $y'(0)$ are known, these can be substituted to solve for the constants.

Analytical Insights and Limitations of the Method

While the method of undetermined coefficients is straightforward and effective for certain types of forcing functions, it has limitations.

Suitability for Specific Forcing Functions

- The method works best when the nonhomogeneous term is a linear combination of functions such as exponentials, sines, cosines, or polynomials.
- For functions outside this set (like e^{x^2} or $\ln x$), alternative methods like variation of parameters are preferable.

Resonance and Overlap in Forcing Function and Homogeneous Solution

- If the forcing function resembles a solution to the homogeneous equation, the initial guess must be modified.
- In this problem, since y_h involves sines and cosines with $\sqrt{7}$, and the forcing function is $\sin x$, which is not a solution to the homogeneous equation, the method proceeds smoothly.

Handling Overlap Cases

- If the forcing term matches the homogeneous solution's form, the particular solution guess must be multiplied by x to account for resonance, e.g., $y_p = x (A \sin x + B \cos x)$.

Conclusion: The Power and Precision of the Method

The method of undetermined coefficients exemplifies a structured approach to solving linear differential equations with specific nonhomogeneous terms. Its implementation in solving $y'' + 7 y = 7 \sin x$ demonstrates how an educated guess, guided by the form of the forcing function, can lead to an explicit particular solution. When combined with the homogeneous solution, it offers a complete description of the system's behavior, subject to initial conditions.

This technique's elegance lies in its simplicity and efficiency, particularly for standard forcing functions. Nonetheless, it requires careful consideration of the nature of the nonhomogeneous term and the potential for resonance. As such, it remains an essential tool in the mathematician's arsenal for tackling differential equations in physics, engineering, and applied mathematics.

In summary, the process involves:

- Solving the homogeneous equation to find \(y_h \).
- Guessing a particular solution \(y_p \) based on the form of the nonhomogeneous term.
- Calculating derivatives and substituting into the original equation.
- Equating coefficients to solve for the undetermined coefficients.
- Combining solutions and applying initial conditions for the complete solution.

The method of undetermined coefficients exemplifies the blend of intuition and algebraic rigor, providing clear pathways to solutions in a wide array of applications.

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