Using The Frobenius Method, Solve The Ordinary Differential Equation 3xy" + (2 - X)y - 2y = 0 . Then

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Solving ordinary differential equations (ODEs) is a fundamental aspect of mathematical analysis, especially when dealing with complex equations that do not lend themselves to straightforward solutions. Among the various methods available, the Frobenius method stands out as a powerful technique for solving linear differential equations near regular singular points. In this article, we will explore how to use the Frobenius method to solve the differential equation:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

and proceed to analyze the solution thoroughly.

Understanding the Differential Equation

Before applying any solution method, it's crucial to understand the structure and nature of the given differential equation:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

Simplifying the equation:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$
$$ 3x y'' + (2 - x - 2) y = 0 $$
$$ 3x y'' + (-x) y = 0 $$
or, more straightforwardly,

$$ 3x y'' - x y = 0 $$

which can be rewritten as:

$$ 3x y'' = x y $$

Dividing both sides by x (assuming $x \neq 0$):

$$ 3 y'' = y $$

This indicates the differential equation, away from $x=0$, simplifies significantly. However, the original form suggests a potential singularity at $x=0$, which prompts us to consider the

Frobenius method, especially around the singular point $x=0$.

Rearranging the Equation into Standard Form

The Frobenius method is typically used for equations that can be expressed as:

$$x^2 y'' + p(x) x y' + q(x) y = 0$$

or similar forms where the point $x=0$ is a regular or irregular singular point.

Let's attempt to rewrite the original differential equation in a form suitable for the Frobenius method.

Starting with the original:

$$3x y'' + (2 - x) y - 2 y = 0$$

which simplifies to:

$$3x y'' + (2 - x - 2) y = 0$$
$$3x y'' - x y = 0$$

Dividing through by x:

$$3 y'' - y = 0$$

which is a second-order linear differential equation with constant coefficients.

Key insight: The differential equation reduces to:

$$3 y'' - y = 0$$

which is a linear homogeneous differential equation with constant coefficients, solvable via characteristic equations.

Solving the Simplified Differential Equation

The characteristic equation for:

$$3 y'' - y = 0$$

is:

$$ 3 r^2 - 1 = 0 $$

Solving for r:

$$ r^2 = \frac{1}{3} $$
$$ r = \pm \frac{1}{\sqrt{3}} $$

Thus, the general solution is:

$$ y(x) = C_1 e^{x / \sqrt{3}} + C_2 e^{-x / \sqrt{3}} $$

where C_1 and C_2 are arbitrary constants.

Considering the Frobenius Method

While the simplified form yields a straightforward solution, the initial steps suggest that the original differential equation has a singular point at $x=0$. To examine this in detail, and to demonstrate the Frobenius method, let's revisit the original form:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

which further simplifies to:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

or equivalently,

$$ 3x y'' + (2 - x - 2) y = 0 $$
$$ 3x y'' - x y = 0 $$

Dividing both sides by x:

$$ 3 y'' - y = 0 $$

This is consistent with our earlier simplified form, confirming that away from $x=0$, the solution involves exponential functions.

However, to demonstrate the Frobenius method, suppose we consider the original differential equation in the form:

$$ y'' + p(x) y' + q(x) y = 0 $$

or equivalently, analyze the behavior near $x=0$ as a regular singular point.

Applying the Frobenius Method Step-by-Step

The Frobenius method involves assuming a solution of the form:

$$ y(x) = x^r \sum_{n=0}^\infty a_n x^n $$

where:

- r is the indicial exponent to be determined.
- $a_0 \neq 0$ is the first coefficient.

To proceed, we need to express the differential equation in standard form:

$$ y'' + p(x) y' + q(x) y = 0 $$

from the original equation.

Expressing the Equation in Standard Form

Starting with:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

which simplifies to:

$$ 3x y'' + (2 - x - 2) y = 0 $$
$$ 3x y'' - x y = 0 $$

Dividing through by $3x$:

$$ y'' - \frac{1}{3} y = 0 $$

Now, the standard form is:

$$ y'' + p(x) y' + q(x) y = 0 $$

but since the equation reduces further to:

$$ y'' = \frac{1}{3} y $$

which is an equation with constant coefficients, the Frobenius method isn't necessary here.

Understanding When to Use the Frobenius Method

The Frobenius method is especially useful when:

- The differential equation has a singular point at $x=0$.
- The point $x=0$ is a regular singular point.
- The equation cannot be simplified into constant coefficients directly.

In our case, the original equation simplifies to a constant-coefficient differential equation after division, indicating that solutions are exponential functions. Therefore, the Frobenius method is not required here.

But, for educational purposes, let's consider a more general form where the Frobenius method would be appropriate:

$$
x^2 y'' + p(x) x y' + q(x) y = 0
$$

with $p(x)$ and $q(x)$ analytic near $x=0$.

General Procedure of Frobenius Method

In scenarios where the Frobenius method is applicable, the steps are:

1. Identify the singular point and check if it is regular or irregular.

2. Assume a solution of the form $y(x) = x^r \sum_{n=0}^\infty a_n x^n$.

3. Compute derivatives $y'(x)$ and $y''(x)$ based on the assumed form.

4. Substitute into the differential equation and group terms by powers of x.

5. Set the coefficient of the lowest power of x to zero to find the indicial equation, which determines r.

6. Solve the recurrence relation for the coefficients a_n.

7. Construct the general solution from the series solutions corresponding to the roots r.

Conclusion

In the specific differential equation:

$$ 3x y'' + (2 - x) y - 2 y = 0 $$

we find that, after simplification, the equation reduces to:

$$ y'' = \frac{1}{3} y $$

which has the general solution:

$$ y(x) = C_1 e^{x/\sqrt{3}} + C_2 e^{-x/\sqrt{3}} $$

This indicates that the problem may not necessitate the Frobenius method for solutions away from $x=0$. However, if the original problem involved more complex functions or singular points, the Frobenius method would be an invaluable tool.

In summary:

- Always analyze the structure of your differential equation to determine the appropriate solution method.
- When dealing with singular points, especially regular singular points, the Frobenius method provides a systematic approach to find power series solutions.
- For equations reducible to constant coefficients, simpler methods such as characteristic equations suffice.

Additional Resources for Learning Differential Equations

- "Elementary Differential Equations and Boundary Value Problems" by Boyce and DiPrima.
- Online tutorials on Frobenius method applications.
- Interactive tools for solving differential equations symbolically.

Final Note: While the initial differential equation simplifies significantly, understanding the Frobenius method's application prepares you

Frequently Asked Questions

What is the first step in applying the Frobenius method to the differential equation 3xy'' + (2 - x)y - 2y = 0?

The first step is to rewrite the differential equation in standard form and identify the singular point, typically at x = 0, then assume a power series solution y = x^r ∑ a_n x^n.

How do you determine the indicial equation when using the Frobenius method for this differential equation?

Substitute the assumed solution y = x^r into the lowest order terms of the differential equation and collect coefficients to form the indicial equation, which helps determine the possible values of r.

What are the steps to find the recurrence relation for the coefficients a_n in the power series solution?

Differentiate the power series as needed, substitute into the differential equation, and equate coefficients of like powers of x to derive a recurrence relation for a_n in terms of previous coefficients.

How do the roots of the indicial equation influence the form of the solution?

The roots determine the nature of the solutions: if roots are distinct and do not differ by an integer, two independent solutions are obtained directly; if roots differ by an integer, special methods are needed, and logarithmic terms may appear.

What challenges might arise when solving the differential equation 3xy'' + (2 - x)y - 2y = 0 using the Frobenius method?

Potential challenges include handling irregular singular points, managing complex recurrence relations, and ensuring convergence of the power series solution near the singularity at x=0.

After finding the power series solution using the Frobenius method, what steps are taken to express the general solution to the differential equation?

Combine the independent solutions obtained from different roots of the indicial equation, and include arbitrary constants to write the general solution as a linear combination of these solutions.

Additional Resources

Using The Frobenius Method, Solve The Ordinary Differential Equation 3xy" + (2 - x)y - 2y = 0

The Frobenius method is a powerful technique for solving linear ordinary differential equations (ODEs), especially those with singular points where standard power series solutions may not apply directly. In this article, we explore the application of the Frobenius method to the differential equation:

$$3x y'' + (2 - x) y' - 2 y = 0.$$

This process involves transforming the equation into a form suitable for Frobenius series expansion, determining the indicial equation, and systematically deriving the solution series. By meticulously following each step, we demonstrate how the Frobenius method can be effectively employed to find solutions to such differential equations.

Understanding the Differential Equation

Before applying the Frobenius method, it’s essential to analyze the form of the differential equation:

$$3x y'' + (2 - x) y' - 2 y = 0.$$

This is a second-order linear ODE with variable coefficients. Notice that the coefficients are polynomial functions of x, which suggests that the Frobenius method could be applicable, especially around points where the coefficients are singular or regular singular points.

Key observations:

- The point $x = 0$ is a candidate for a regular singular point because the coefficients involve x in the denominator if the equation is rearranged.
- To verify whether $x=0$ is a singular point, rewrite the equation in standard form.

Rearranging to Standard Form

The standard form for Frobenius method application is:

$$y'' + p(x) y' + q(x) y = 0.$$

Divide the entire original equation by $3x$:

$$
y'' + \frac{2 - x}{3x} y' - \frac{2}{3x} y = 0.
$$

Now, identify:

- $p(x) = \frac{2 - x}{3x}$,
- $q(x) = - \frac{2}{3x}$.

At $x=0$, both $p(x)$ and $q(x)$ involve $1/x$, indicating that $x=0$ is a regular singular point. Therefore, the Frobenius method is suitable for finding solutions around $x=0$.

Applying the Frobenius Method

The Frobenius method assumes a solution of the form:

$$
y(x) = x^{r} \sum_{n=0}^{\infty} a_n x^{n},
$$

where:

- r is the indicial exponent to be determined,
- $a_0 \neq 0$.

Our goal is to find r and the coefficients a_n.

Step 1: Compute y' and y''

Given:

$$
y = x^{r} \sum_{n=0}^{\infty} a_n x^{n},
$$

then:

$$
y' = \frac{d}{dx} \left(x^{r} \sum_{n=0}^{\infty} a_n x^{n} \right) = x^{r-1} \sum_{n=0}^{\infty} a_n (r + n) x^{n}.
$$

Similarly,

$$
y'' = \frac{d}{dx} y' = x^{r-2} \sum_{n=0}^{\infty} a_n (r + n)(r + n -1) x^{n}.
$$

Step 2: Substitute into the differential equation

Recall the rearranged form:

$$
y'' + p(x) y' + q(x) y = 0,
$$

with

$$
p(x) = \frac{2 - x}{3x}, \quad q(x) = - \frac{2}{3x}.
$$

Substitute y, y', y'':

$$
x^{r-2} \sum_{n=0}^\infty a_n (r + n)(r + n -1) x^{n} + \frac{2 - x}{3x} \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n} - \frac{2}{3x} \times x^{r} \sum_{n=0}^\infty a_n x^{n} = 0.
$$

Simplify each term:

1. y'':

$$
x^{r-2} \sum_{n=0}^\infty a_n (r + n)(r + n - 1) x^{n}.
$$

2. $p(x) y'$:

$$
\left(\frac{2 - x}{3x} \right) \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n} = \frac{2 - x}{3x} \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n}.
$$

3. $q(x) y$:

$$
- \frac{2}{3x} \times x^{r} \sum_{n=0}^\infty a_n x^{n} = - \frac{2}{3x} \times x^{r} \sum_{n=0}^\infty a_n x^{n}.
$$

Step 3: Simplify and write in powers of x

Express each term as a power series in x:

- First term:

$$
x^{r-2} \sum_{n=0}^\infty a_n (r + n)(r + n - 1) x^{n} = \sum_{n=0}^\infty a_n (r + n)(r + n - 1) x^{n + r - 2}.
$$

- Second term:

$$
\frac{2 - x}{3x} \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n} = \frac{2 - x}{3x} \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n}.
$$

Rewrite numerator:

$$
(2 - x) = 2 - x.
$$

Divide by $3x$:

$$
\frac{2 - x}{3x} = \frac{2}{3x} - \frac{1}{3}.
$$

Thus, the second term becomes:

$$
\left(\frac{2}{3x} - \frac{1}{3} \right) \times x^{r-1} \sum_{n=0}^\infty a_n (r + n) x^{n} = \left(\frac{2}{3x} \times x^{r-1} \sum a_n (r + n) x^{n} \right) - \frac{1}{3} \times x^{r-1} \sum a_n (r + n) x^{n}.
$$

Simplify each:

- $\frac{2}{3x} \times x^{r-1} \sum a_n (r + n) x^{n} = \frac{2}{3} x^{r-2} \sum a_n (r + n) x^{n}$.
- $- \frac{1}{3} \times x^{r-1} \sum a_n (r + n) x^{n} = - \frac{1}{3} x^{r-1} \sum a_n (r + n) x^{n}$.

- Third term:

$$
- \frac{2}{3x} \times x^{r} \sum a_n x^{n} = - \frac{2}{3} x^{r-1} \sum a_n x^{n}.
$$

Step 4: Combine all terms

Express all sums with a common power $x^{n + r - 2}$:

- First sum:

$$
\sum_{n=0}^\infty a_n (r + n)(r + n - 1) x^{n + r - 2}.
$$

- Second sum:

$$
\frac{2}{3} \sum_{n=0}^\infty a_n (r + n) x^{n + r - 2} - \frac{1}{3} \sum_{n=0}^\infty a_n (r + n) x^{n + r - 1}.
$$

- Third sum:

$$
- \frac{2}{3} \sum_{n=0}^\infty a_n x^{n + r - 1}.
$$

Now, to combine all sums, re-index the sums involving \(x^{n + r

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