Using The Laplace Transform, Solve The IVPy = 5y1-4y2-9t2+2t, Y2 = 10y1-772-172-2t, 31(0) = 3, Yz(0)

Using The Laplace Transform, Solve The IVPy = 5y1-4y2-9t2+2t, Y2 = 10y1-772-172-2t, 31(0) = 3, Yz(0)

Solving systems of differential equations is a fundamental aspect of mathematical analysis, especially in engineering, physics, and applied sciences. The Laplace transform offers a powerful technique to convert complex differential equations into manageable algebraic equations. In this article, we will explore step-by-step how to use the Laplace transform to solve a system of differential equations, specifically focusing on the problem:

$$ \text{IVP:} \quad y' = 5y_1 - 4y_2 - 9t^2 + 2t $$
$$ \text{and} \quad y_2 = 10y_1 - 772 - 172 - 2t $$
with initial conditions:
$$ y_1(0) = 3, \quad y_2(0) = \text{(to be specified)} $$

This comprehensive guide aims to clarify each step, from understanding the problem to applying the Laplace transform and finally obtaining the solution. Whether you're a student learning differential equations or a professional applying these techniques, this article provides a detailed, structured approach.

Understanding the Problem and Its Components

Before diving into the solution process, it’s crucial to interpret the problem correctly and identify the key components involved.

1. The Differential Equation

The primary differential equation provided is:

$$ y' = 5 y_1 - 4 y_2 - 9 t^2 + 2 t $$

Here, y' typically denotes the derivative of a function $y(t)$. The notation suggests that $y_1(t)$ and $y_2(t)$ are functions related to the system, possibly representing different state variables.

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2. The Relationship between y_1 and y_2

The second relation is:

$$ y_2 = 10 y_1 - 772 - 172 - 2 t $$

which simplifies to:

$$ y_2 = 10 y_1 - 944 - 2 t $$

This indicates a linear relationship between the two functions, which can be leveraged to reduce the system to a single equation or to express one variable in terms of the other.

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3. Initial Conditions

The initial condition is provided as:

$$ y_1(0) = 3 $$

However, the initial condition for $y_2(0)$ appears incomplete or possibly misprinted in the original prompt. For a complete solution, an initial condition like $y_2(0) = y_{2,0}$ is essential. Assuming this is a typo, and the intended initial condition is $y_2(0) = y_{2,0}$.

Note: If the initial condition for $y_2(0)$ is missing, it's necessary to clarify or deduce it from context.

Step-by-Step Solution Method Using Laplace Transform

Applying the Laplace transform involves transforming differential equations from the time domain into the complex frequency domain, solving algebraically, and then inverse transforming back to obtain the solution in the time domain.

1. Recall the Laplace Transform Basics

The Laplace transform of a function $f(t)$ is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-s t} f(t) dt$$

For derivatives:

$$\mathcal{L}\{f'(t)\} = s F(s) - f(0)$$

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2. Express the Differential Equation in Terms of Laplace Transforms

Let:

$$\mathcal{L}\{ y_1(t) \} = Y_1(s)$$
$$\mathcal{L}\{ y_2(t) \} = Y_2(s)$$

Applying the Laplace transform to the differential equation:

$$y' = 5 y_1 - 4 y_2 - 9 t^2 + 2 t$$

becomes:

$$s Y(s) - y(0) = 5 Y_1(s) - 4 Y_2(s) - 9 \mathcal{L}\{ t^2 \} + 2 \mathcal{L}\{ t \}$$

Similarly, the second relation:

$$y_2 = 10 y_1 - 944 - 2 t$$

can be transformed as needed.

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3. Compute the Laplace Transforms of Non-Homogeneous Terms

Key transforms:

- $\mathcal{L}\{ t \} = \frac{1}{s^2}$
- $\mathcal{L}\{ t^2 \} = \frac{2}{s^3}$

Substituting:

$$ s Y(s) - y(0) = 5 Y_1(s) - 4 Y_2(s) - 9 \times \frac{2}{s^3} + 2 \times \frac{1}{s^2} $$

which simplifies to:

$$ s Y(s) - y(0) = 5 Y_1(s) - 4 Y_2(s) - \frac{18}{s^3} + \frac{2}{s^2} $$

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4. Express $y_2(t)$ in Terms of $y_1(t)$ and Transform

From the relation:

$$ y_2 = 10 y_1 - 944 - 2 t $$

taking Laplace transform:

$$ Y_2(s) = 10 Y_1(s) - 944 \cdot \frac{1}{s} - 2 \times \frac{1}{s^2} $$

since $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{ t \} = \frac{1}{s^2}$.

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5. Formulate and Solve the System in the Laplace Domain

Substituting $Y_2(s)$ into the transformed differential equation:

$$ s Y_1(s) - y_1(0) = 5 Y_1(s) - 4 \left(10 Y_1(s) - 944 \frac{1}{s} - 2 \frac{1}{s^2} \right) - \frac{18}{s^3} + \frac{2}{s^2} $$

Simplify:

$$ s Y_1(s) - 3 = 5 Y_1(s) - 40 Y_1(s) + \frac{4 \times 944}{s} + \frac{8}{s^2} - \frac{18}{s^3} + \frac{2}{s^2} $$

which becomes:

$$ s Y_1(s) - 3 = -35 Y_1(s) + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3} $$

Bring all $Y_1(s)$ terms to one side:

$$ s Y_1(s) + 35 Y_1(s) = 3 + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3} $$

Factor:

$$ (s + 35) Y_1(s) = 3 + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3} $$

Express the right side over a common denominator s^3:

$$ (s + 35) Y_1(s) = \frac{3 s^3 + 3776 s^2 + 10 s - 18}{s^3} $$

Finally, solve for $Y_1(s)$:

$$ Y_1(s) = \frac{3 s^3 + 3776 s^2 + 10 s - 18}{s^3 (s + 35)} $$

Inverse Laplace Transform and Final Solution

To find $y_1(t)$, perform the inverse Laplace transform of $Y_1(s)$. This involves:

- Partial fraction decomposition
- Recognizing standard inverse transforms
- Combining results to express $y_1(t)$

Similarly, $y_2(t)$ can be recovered via:

$$ y_2(t) = 10 y_1(t) - 944 - 2 t $$

Practical Steps for the Inversion Process

1. Decompose $Y_1(s)$ into partial fractions

Given the complex numerator and denominator, break $Y_1(s)$ into simpler fractions:

- Express as sums of terms like $\frac{A}{s}$, $\frac{B}{s^2}$

Frequently Asked Questions

What is the first step in solving the given system of differential equations using Laplace transforms?

The first step is to take the Laplace transform of each differential equation, applying initial conditions to convert the system into algebraic equations in the Laplace domain.

How do you handle the initial conditions Y1(0) = 3 and Y2(0) in the Laplace transform process?

Initial conditions are incorporated by replacing the derivatives with sY1(s) - Y1(0) and sY2(s) - Y2(0) in the Laplace domain equations, ensuring the initial values are considered during transformation.

What are the main challenges when solving the system of equations after applying the Laplace transform?

The main challenges include solving the resulting algebraic system for Y1(s) and Y2(s), managing the coupled equations, and then applying the inverse Laplace transform to find the functions y1(t) and y2(t).

How do you perform the inverse Laplace transform to find y1(t) and y2(t) after solving for Y1(s) and Y2(s)?

You decompose the algebraic expressions into simpler fractions using partial fraction decomposition, then apply known inverse Laplace transform formulas to find y1(t) and y2(t).

What is the significance of correctly applying initial conditions in the Laplace transform method for this system?

Applying initial conditions accurately ensures the solution reflects the specific behavior of the system at t=0, leading to the correct particular solution rather than a general one.

Can you summarize the overall procedure to solve the system of differential equations using the Laplace transform?

Yes, the steps are: 1) take the Laplace transform of each equation; 2) incorporate initial conditions; 3) solve the resulting algebraic system for Y1(s) and Y2(s); 4) perform inverse Laplace transforms to find y1(t) and y2(t).

Additional Resources

Using the Laplace Transform to Solve the Initial Value Problem $y''_p = 5 y_1 - 4 y_2 - 9 t^2 + 2 t$, $y_2 = 10 y_1 - 772 - 172 - 2 t$, with Initial Conditions $y_1(0) = 3$, $y_2(0)$

Introduction

The Laplace transform is an integral transform that converts differential equations from the time domain into an algebraic problem in the complex frequency domain. It is a powerful tool in solving linear ordinary differential equations (ODEs), especially when initial conditions are specified. This review explores the process of solving a coupled second-order initial value problem (IVP) using the Laplace transform, with detailed steps, explanations, and insights into the underlying principles.

Understanding the Problem Statement

The problem involves a system of second-order differential equations with initial conditions:

- $y''_p(t) = 5 y_1(t) - 4 y_2(t) - 9 t^2 + 2 t$
- $y_2(t) = 10 y_1(t) - 772 - 172 - 2 t$
- Initial conditions: $y_1(0) = 3$, $y_2(0) =$ (not explicitly provided, but critical for the full solution)

At first glance, the notation appears somewhat inconsistent or possibly contains typographical errors. For clarity, assume the following:

- The system involves two functions $y_1(t)$ and $y_2(t)$.
- $y_1(t)$ and $y_2(t)$ are coupled via differential and algebraic relations.
- The derivatives are with respect to t.

The goal is to find explicit expressions for $y_1(t)$ and $y_2(t)$ using the Laplace transform method.

Step 1: Clarify the System and Notation

Before applying the Laplace transform, it is essential to interpret the equations correctly:

- The first differential equation involves $y''_p(t)$. The subscript p may be a notation for the particular solution or a typo; assume it refers to $y_1(t)$. So, rewrite as:

\[
y_1''(t) = 5 y_1(t) - 4 y_2(t) - 9 t^2 + 2 t
\]

- The second equation relates \(y_2(t) \) to \(y_1(t) \):

\[
y_2(t) = 10 y_1(t) - 772 - 172 - 2 t
\]

Simplify the constants:

\[
y_2(t) = 10 y_1(t) - 944 - 2 t
\]

- Initial conditions:

\[
y_1(0) = 3, \quad y_2(0) = \text{unknown}
\]

Given this, the problem reduces to solving a coupled system with one second-order differential equation and an algebraic relation.

Step 2: Formulate the System

The key is to express everything in terms of \(y_1(t) \), as \(y_2(t) \) can be written in terms of \(y_1(t) \):

\[
y_2(t) = 10 y_1(t) - 944 - 2 t
\]

Differentiate \(y_2(t) \):

\[
y_2'(t) = 10 y_1'(t) - 2
\]

And differentiate again:

\[

$$
y_2''(t) = 10 y_1''(t)
$$

But $y_2''(t)$ is not directly given; instead, the main differential equation involves $y_1''(t)$.

Step 3: Applying the Laplace Transform

The Laplace transform of a function $f(t)$ is defined as:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt
$$

Key properties used in solving differential equations:

- Linearity: $\mathcal{L}\{a f(t) + b g(t)\} = a F(s) + b G(s)$
- Derivative: $\mathcal{L}\{f'(t)\} = s F(s) - f(0)$
- Second derivative: $\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$

Step 4: Transform the Differential Equation

Transform the main equation:

$$
y_1''(t) = 5 y_1(t) - 4 y_2(t) - 9 t^2 + 2 t
$$

Applying the Laplace transform:

$$
\mathcal{L}\{ y_1''(t) \} = s^2 Y_1(s) - s y_1(0) - y_1'(0)
$$

Similarly, for the right side:

$$
\mathcal{L}\{ 5 y_1(t) \} = 5 Y_1(s)
$$

$$
\mathcal{L}\{ -4 y_2(t) \} = -4 Y_2(s)
$$

$$
\mathcal{L}\{ -9 t^2 \} = -9 \frac{2}{s^3} = -\frac{18}{s^3}
$$

$$
\mathcal{L}\{ 2 t \} = 2 \frac{1}{s^2} = \frac{2}{s^2}
$$

Note: The Laplace transform of t^n is $n! / s^{n+1}$.

Assuming initial conditions:

$$
y_1(0) = 3
$$

and

$$
y_1'(0) = \text{unknown}
$$

Suppose $y_1'(0) = a$. For initial value problems, the derivative at zero is often specified; if not, it remains as a parameter.

Now, the transformed differential equation becomes:

$$
s^2 Y_1(s) - s \cdot 3 - a = 5 Y_1(s) - 4 Y_2(s) - \frac{18}{s^3} + \frac{2}{s^2}
$$

Rearranged:

$$
s^2 Y_1(s) - 3 s - a = 5 Y_1(s) - 4 Y_2(s) - \frac{18}{s^3} + \frac{2}{s^2}
$$

Step 5: Express $Y_2(s)$ in Terms of $Y_1(s)$

From the algebraic relation:

$$
y_2(t) = 10 y_1(t) - 944 - 2 t
$$

Applying Laplace:

$$
Y_2(s) = 10 Y_1(s) - 944 \frac{1}{s} - 2 \cdot \frac{1}{s^2}
$$

This expression allows substitution back into the main equation, leading to an algebraic equation solely in $Y_1(s)$:

$$
s^2 Y_1(s) - 3 s - a = 5 Y_1(s) - 4 \left(10 Y_1(s) - 944 \frac{1}{s} - 2 \frac{1}{s^2} \right) - \frac{18}{s^3} + \frac{2}{s^2}
$$

Simplify the right side:

$$
5 Y_1(s) - 40 Y_1(s) + \frac{4 \cdot 944}{s} + \frac{8}{s^2} - \frac{18}{s^3} + \frac{2}{s^2}
$$

$$
= (5 - 40) Y_1(s) + \frac{3776}{s} + \left(\frac{8}{s^2} + \frac{2}{s^2} \right) - \frac{18}{s^3}
$$

$$
= -35 Y_1(s) + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3}
$$

Now, collect all terms:

$$
s^2 Y_1(s) - 3 s - a = -35 Y_1(s) + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3}
$$

Bring all $Y_1(s)$ terms to one side:

\[
s^2 Y_1(s) + 35 Y_1(s) = 3 s + a + \frac{3776}{s} + \frac{10}{s^2} - \frac{18}{s^3}
\]

Factor:

\[
Y_1

Using The Laplace Transform Solve The Ivpy 5y1 4y2 9t2 2t Y2 10y1 772 172 2t 31 0 3 Yz 0

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