Using The Laplace Transform, Solve: Y" +9y=r(t), Y(0) = 0, Y'(0) = 10, Where R(t) = 8 Sint If 0 <

Using The Laplace Transform, Solve: Y" +9y=r(t), Y(0) = 0, Y'(0) = 10, Where R(t) = 8 Sint If 0 <

The Laplace transform is a powerful mathematical tool widely used to solve linear ordinary differential equations (ODEs), especially those with initial conditions. In this article, we will explore how to apply the Laplace transform to solve the second-order differential equation:

$$ Y'' + 9y = r(t), \quad Y(0) = 0, \quad Y'(0) = 10, \quad \text{where} \quad r(t) = 8 \sin t \quad \text{for} \quad t > 0. $$

This problem showcases the effectiveness of Laplace transforms in handling differential equations with sinusoidal forcing functions and initial conditions, providing a systematic approach to find the solution $y(t)$.

Understanding the Differential Equation and Initial Conditions

Before delving into the solution process, it is essential to comprehend the structure of the differential equation and the initial conditions.

The Differential Equation

The equation:

$$ Y'' + 9y = r(t) $$

is a nonhomogeneous second-order linear differential equation with constant coefficients. The term $r(t) = 8 \sin t$ acts as an external forcing function, often representing an oscillatory input in physical systems such as mechanical vibrations or electrical circuits.

Initial Conditions

Given:

$$ Y(0) = 0, \quad Y'(0) = 10, $$

these specify the initial state of the system at $t = 0$. Incorporating initial conditions is crucial for obtaining a unique solution.

Applying the Laplace Transform to the Differential Equation

The core idea behind using the Laplace transform is to convert the differential equation from the time domain into the algebraic domain, where it becomes easier to solve.

The Laplace Transform of Derivatives

Recall the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ has the following properties for derivatives:

- $\mathcal{L}\{Y'(t)\} = sY(s) - Y(0)$
- $\mathcal{L}\{Y''(t)\} = s^2Y(s) - sY(0) - Y'(0)$

Using these, we can transform the differential equation.

Transforming the Equation

Applying the Laplace transform to both sides:

$$
\mathcal{L}\{Y'' + 9y\} = \mathcal{L}\{r(t)\}
$$

which becomes:

$$
s^2 Y(s) - sY(0) - Y'(0) + 9 Y(s) = \mathcal{L}\{8 \sin t\}
$$

Substituting the initial conditions:

$$
s^2 Y(s) - 0 - 10 + 9 Y(s) = \mathcal{L}\{8 \sin t\}
$$

Simplifies to:

$$
(s^2 + 9) Y(s) - 10 = 8 \mathcal{L}\{\sin t\}
$$

Calculating the Laplace Transform of $r(t) = 8 \sin t$

The Laplace transform of $\sin at$ is a standard result:

$$
\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}
$$

For $a = 1$:

$$
\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}
$$

Therefore:

$$
\mathcal{L}\{8 \sin t\} = 8 \times \frac{1}{s^2 + 1} = \frac{8}{s^2 + 1}
$$

Solving for $Y(s)$ in the s-Domain

Substituting the Laplace transform of $r(t)$ into the transformed differential equation:

$$
(s^2 + 9) Y(s) - 10 = \frac{8}{s^2 + 1}
$$

Rearranged to solve for $Y(s)$:

$$
Y(s) = \frac{\frac{8}{s^2 + 1} + 10}{s^2 + 9}
$$

Expressed as a single fraction:

$$
Y(s) = \frac{8}{(s^2 + 1)(s^2 + 9)} + \frac{10}{s^2 + 9}
$$

Now, the goal is to find the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Partial Fraction Decomposition and Inverse Laplace Transform

To facilitate the inverse Laplace transform, we decompose the first term into partial fractions.

Decomposing $\frac{8}{(s^2 + 1)(s^2 + 9)}$

Assuming:

$$
\frac{8}{(s^2 + 1)(s^2 + 9)} = \frac{A}{s^2 + 1} + \frac{B}{s^2 + 9}
$$

Multiply both sides by $(s^2 + 1)(s^2 + 9)$:

$$
8 = A(s^2 + 9) + B(s^2 + 1)
$$

Expanding:

$$
8 = A s^2 + 9A + B s^2 + B
$$

Combine like terms:

$$
8 = (A + B) s^2 + (9A + B)
$$

Since this holds for all s, equate coefficients:

- Coefficient of s^2:

$$ A + B = 0 \Rightarrow B = -A $$

- Constant term:

$$ 9A + B = 8 $$

Substitute $B = -A$:

$$
9A - A = 8 \Rightarrow 8A = 8 \Rightarrow A = 1
$$

Therefore:

$$
B = -1
$$

Hence:

$$
\frac{8}{(s^2 + 1)(s^2 + 9)} = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9}
$$

Expressing $Y(s)$ as a sum of simpler fractions

Now, $Y(s)$ becomes:

$$
Y(s) = \left(\frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} \right) + \frac{10}{s^2 + 9}
$$

Group similar terms:

$$
Y(s) = \frac{1}{s^2 + 1} + \left(-\frac{1}{s^2 + 9} + \frac{10}{s^2 + 9} \right) = \frac{1}{s^2 + 1} + \frac{9}{s^2 + 9}
$$

Finding the Inverse Laplace Transform to Get \(

y(t) \)

Using the standard inverse Laplace transforms:

$$
\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \sin a t
$$

we obtain:

$$
y(t) = \sin t + 9 \sin 3t
$$

Final Solution and Interpretation

The solution to the differential equation:

$$
Y'' + 9 y = 8 \sin t
$$

with initial conditions $Y(0) = 0$ and $Y'(0) = 10$ is:

$$
\boxed{
y(t) = \sin t + 9 \sin 3t
}
$$

for $t > 0$.

This solution combines the homogeneous response (which would involve exponential functions if solved directly) with the particular solution driven by the sinusoidal input $8 \sin t$. The approach via Laplace transforms simplifies the process, especially when initial conditions are involved.

Advantages of Using the Laplace Transform in Differential Equations

Employing Laplace transforms offers several benefits:

- **Simplification of differential equations:** Converts differential equations into algebraic equations.
- **Handling initial conditions:** Incorporates initial values directly into the transformation process.
- **Dealing with complex forcing functions:** Facilitates solving equations with sinusoidal, exponential, or piecewise functions.
- **Systematic approach:** Provides a straightforward method to obtain particular and homogeneous solutions.

Conclusion

Using the Laplace transform to solve differential equations like \(Y'' + 9 y =

Frequently Asked Questions

What is the initial step in solving the differential equation Y'' + 9Y = r(t) using Laplace transforms?

The initial step is to take the Laplace transform of both sides of the differential equation, incorporating the initial conditions, which simplifies the equation into an algebraic form in the Laplace domain.

How do you handle the Laplace transform of the forcing function r(t) = 8 sin(t)?

The Laplace transform of $8 \sin(t)$ is $8 (s / (s^2 + 1))$, which is used directly in the algebraic equation after transforming the differential equation.

What is the Laplace transform of the second derivative Y'' given the initial conditions?

Using the initial conditions $Y(0)=0$ and $Y'(0)=10$, the Laplace transform of Y'' is $s^2 Y(s) - s Y(0) - Y'(0)$, which simplifies to $s^2 Y(s) - 10$.

After transforming the differential equation, what algebraic equation do you solve for Y(s)?

The algebraic equation is (s^2 + 9) Y(s) = 8 (s / (s^2 + 1)) + 10, which can be rearranged to Y(s) = [8 (s / (s^2 + 1)) + 10] / (s^2 + 9).

How do you find the inverse Laplace transform to obtain the solution y(t)?

You decompose Y(s) into simpler parts using partial fractions and known Laplace transform pairs, then apply the inverse Laplace transform to each part to find y(t).

What is the significance of the initial conditions in solving this differential equation?

The initial conditions Y(0)=0 and Y'(0)=10 determine the specific solution to the differential equation, affecting the particular form of Y(s) and consequently y(t) after inverse transformation.

Can you summarize the overall process to solve Y'' + 9Y = 8 sin(t) with given initial conditions using Laplace transforms?

Yes, the process involves taking the Laplace transform of both sides, incorporating initial conditions, solving for Y(s), performing partial fraction decomposition if necessary, and then applying the inverse Laplace transform to find y(t).

Additional Resources

Using The Laplace Transform to solve differential equations is a powerful technique extensively used in engineering, physics, and applied mathematics. It simplifies complex differential equations by converting them from the time domain into the s-domain (complex frequency domain), where algebraic methods can be employed to find solutions more straightforwardly. This approach is especially advantageous when dealing with initial value problems involving non-homogeneous terms, such as forcing functions, impulses, or sinusoidal inputs. In this article, we will explore how to apply the Laplace Transform to solve the second-order differential equation:

$$ Y'' + 9Y = r(t), \quad \text{with} \quad Y(0) = 0, \quad Y'(0) = 10, $$

where

$$ r(t) = 8 \sin t, \quad t > 0. $$

We will walk through each step meticulously, highlighting the theoretical foundation, the transformation process, algebraic solution, and the inverse Laplace Transform to obtain the explicit solution $Y(t)$. Additionally, we will discuss key features, advantages, and limitations of the method.

Understanding the Differential Equation and Initial Conditions

Before delving into the Laplace Transform process, it is essential to understand the problem's structure and the physical or mathematical context. The differential equation:

$$Y'' + 9Y = r(t)$$

is a standard second-order linear non-homogeneous differential equation. The term $9Y$ indicates a simple harmonic oscillator with natural frequency $\omega = 3$, since $\omega^2 = 9$. The forcing function $r(t) = 8 \sin t$ introduces an external sinusoidal input, which could model, for example, a driven oscillatory system.

The initial conditions:

$$Y(0) = 0, \quad Y'(0) = 10$$

set the initial displacement and velocity, respectively. These are crucial for determining the particular solution uniquely.

Why Use the Laplace Transform?

Applying the Laplace Transform in this context offers several advantages:

- Simplification of Derivatives: Transforms derivatives into algebraic expressions involving the complex frequency s.
- Handling Initial Conditions Naturally: Incorporates initial conditions directly into the algebraic equations.
- Ease of Solving Non-Homogeneous Equations: Converts the differential equation into an algebraic equation in the s-domain.
- Facilitates Dealing with Sinusoidal Forcing Functions: Uses known Laplace Transforms of common functions like sine and cosine.

However, some limitations include:

- Complexity for Non-Standard Functions: Functions not having standard Laplace Transforms may require additional work.
- Inverse Transform Challenges: Sometimes, the inverse Laplace Transform can be complicated, requiring partial fraction decomposition or special techniques.

Step-by-Step Solution Using the Laplace Transform

1. Taking the Laplace Transform of Both Sides

Apply the Laplace Transform $\mathcal{L}\{ \cdot \}$ to both sides:

$$
\mathcal{L}\{Y''\} + 9 \mathcal{L}\{Y\} = \mathcal{L}\{r(t)\}
$$

Recall the Laplace Transform properties:

$$
\mathcal{L}\{Y''(t)\} = s^2 Y(s) - s Y(0) - Y'(0)
$$

$$
\mathcal{L}\{Y(t)\} = Y(s)
$$

$$
\mathcal{L}\{r(t)\} = \mathcal{L}\{8 \sin t\}
$$

Using the initial conditions $Y(0) = 0$ and $Y'(0) = 10$, we get:

$$
s^2 Y(s) - s \times 0 - 10 + 9Y(s) = \mathcal{L}\{8 \sin t\}
$$

which simplifies to:

$$
(s^2 + 9) Y(s) - 10 = \mathcal{L}\{8 \sin t\}
$$

---

2. Computing the Laplace Transform of $r(t) = 8 \sin t$

Standard Laplace Transform:

$$
\mathcal{L}\{\sin a t\} = \frac{a}{s^2 + a^2}
$$

For $a = 1$:

$$
\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}
$$

Therefore:

$$
\mathcal{L}\{8 \sin t\} = 8 \times \frac{1}{s^2 + 1} = \frac{8}{s^2 + 1}
$$

---

3. Formulating the Algebraic Equation for $Y(s)$

Substitute into the transformed equation:

$$
(s^2 + 9) Y(s) - 10 = \frac{8}{s^2 + 1}
$$

Solve for $Y(s)$:

$$
Y(s) = \frac{10}{s^2 + 9} + \frac{8}{(s^2 + 1)(s^2 + 9)}
$$

---

4. Inverse Laplace Transform to Find $Y(t)$

To find $Y(t)$, we need to compute:

$$
Y(t) = \mathcal{L}^{-1} \left\{ \frac{10}{s^2 + 9} \right\} +
\mathcal{L}^{-1} \left\{ \frac{8}{(s^2 + 1)(s^2 + 9)} \right\}
$$

\]

Let's analyze each term separately.

Finding the Inverse Laplace Transforms

Term 1: \(\frac{10}{s^2 + 9} \)

Recognize the standard inverse:

\[
\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a}
\]

Thus:

\[
\mathcal{L}^{-1}\left\{\frac{10}{s^2 + 9}\right\} = 10 \times \frac{\sin 3t}{3} = \frac{10}{3} \sin 3t
\]

Term 2: \(\frac{8}{(s^2 + 1)(s^2 + 9)} \)

This is more complex and requires partial fraction decomposition.

Set:

\[
\frac{8}{(s^2 + 1)(s^2 + 9)} = \frac{A}{s^2 + 1} + \frac{B}{s^2 + 9}
\]

Multiply both sides by \((s^2 + 1)(s^2 + 9) \):

\[
8 = A(s^2 + 9) + B(s^2 + 1)
\]

Set \(s^2 = -1 \):

\[
8 = A(-1 + 9) + B(0) \Rightarrow 8 = A(8) \Rightarrow A = 1

Set $s^2 = -9$:

$$
8 = A(0) + B(-9 + 1) \Rightarrow 8 = B(-8) \Rightarrow B = -1
$$

Therefore,

$$
\frac{8}{(s^2 + 1)(s^2 + 9)} = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9}
$$

Now, take inverse Laplace transforms:

$$
\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a}
$$

So,

$$
\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t
$$

$$
\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 9}\right\} = \frac{\sin 3t}{3}
$$

Thus,

$$
\mathcal{L}^{-1}\left\{\frac{8}{(s^2 + 1)(s^2 + 9)}\right\} = \sin t - \frac{\sin 3t}{3}
$$

Final Expression for $Y(t)$

Combine the two parts:

$$
Y(t) = \frac{10}{3} \sin 3t + 8 \left(\sin t - \frac{\sin 3t}{3} \right)
$$

Simplify:

```
\[
Y(t) = \frac{10}{3} \sin 3t + 8 \sin t - \frac{8}{3} \sin 3t
\]

Group terms involving \( \sin 3t \):

\[
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