Verify That (0, 0) Is A Critical Point, Show That The System Is Locally Linear, And Discuss The Type

Verify That (0, 0) Is A Critical Point, Show That The System Is Locally Linear, And Discuss The Type

Understanding the behavior of dynamical systems near equilibrium points is a fundamental aspect of differential equations and dynamical systems analysis. Among these points, the critical point at the origin, (0, 0), often serves as a pivotal reference for stability and system behavior. This article explores the process of verifying whether (0, 0) is a critical point, demonstrates how to linearize the system locally around this point, and discusses how to classify the nature of this equilibrium—whether it is stable, unstable, or a saddle point.

Verifying That (0, 0) Is a Critical Point

The first step in analyzing a dynamical system is to determine whether a specific point, such as (0, 0), is a critical or equilibrium point. Critical points are states where the system does not change over time, meaning the derivatives of the system's variables are zero at that point.

Definition of a Critical Point

In a system of differential equations:

$$
\begin{cases}
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y)
\end{cases}
$$

a point (x_0, y_0) is a critical point if:

$$
f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0
$$

This means the system remains at (x_0, y_0) if initialized there.

Procedure for Verification

To verify whether (0, 0) is a critical point:

1. Identify the Functions: Write down the functions \(f(x, y)\) and \(g(x, y)\) defining the system.

2. Evaluate at the Origin: Compute \(f(0, 0)\) and \(g(0, 0)\).

3. Check Conditions: Confirm if both are zero.

Example:
Suppose the system is:

\[
\begin{cases}
\frac{dx}{dt} = x - y \\
\frac{dy}{dt} = x + y
\end{cases}
\]

At \((0, 0)\):

- \(f(0, 0) = 0 - 0 = 0\)
- \(g(0, 0) = 0 + 0 = 0\)

Since both are zero, (0, 0) is a critical point.

Showing That The System Is Locally Linear

Once the critical point is identified, the next step is to analyze the system's behavior near this point. Local linearization involves approximating the nonlinear system by its linear counterpart in a small neighborhood around the equilibrium.

What Is Local Linearity?

A system is locally linear at a critical point if, within a small region around that point, the nonlinear functions \(f(x, y)\) and \(g(x, y)\) can be approximated by their first-order Taylor expansions:

\[
\begin{cases}
f(x, y) \approx f(0, 0) + \frac{\partial f}{\partial x}\big|_{(0, 0)} x + \frac{\partial f}{\partial y}\big|_{(0, 0)} y \\
g(x, y) \approx g(0, 0) + \frac{\partial g}{\partial x}\big|_{(0, 0)} x +

\frac{\partial g}{\partial y}\big|_{(0, 0)} y
\end{cases}
\]

Given that \((0, 0)\) is an equilibrium, \(f(0, 0) = 0\) and \(g(0, 0) = 0\), so the linearized system simplifies to:

\[
\begin{cases}
\frac{dx}{dt} = a_{11} x + a_{12} y \\
\frac{dy}{dt} = a_{21} x + a_{22} y
\end{cases}
\]

where

\[
a_{ij} = \frac{\partial \text{(function)}}{\partial \text{(variable)}}
\big|_{(0, 0)}
\]

Constructing the Jacobian Matrix

The key tool for linearization is the Jacobian matrix \(J\):

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\]

Evaluating this at (0, 0):

\[
J(0, 0) = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]

Example:
Using the previous system:

\[
f(x, y) = x - y \Rightarrow \frac{\partial f}{\partial x} = 1, \quad
\frac{\partial f}{\partial y} = -1
\]
\[
g(x, y) = x + y \Rightarrow \frac{\partial g}{\partial x} = 1, \quad

\frac{\partial g}{\partial y} = 1
\]

Thus,

\[
J(0, 0) = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]

This linear approximation provides a simplified model of the system's behavior near (0, 0).

Classifying the Critical Point: The Type

Knowing that (0, 0) is a critical point and having linearized the system, the next step is to determine the nature of this equilibrium: whether it is stable, unstable, a saddle, or otherwise.

Eigenvalues and Their Role

The classification hinges on the eigenvalues of the Jacobian matrix:

\[
\det(J - \lambda I) = 0
\]

where \(I\) is the identity matrix, and \(\lambda\) represents the eigenvalues.

- The sign and nature of these eigenvalues dictate the type of critical point:

Eigenvalues	Type of Critical Point	Behavior
Real, both negative	Stable node	Trajectories approach (0, 0)
Real, both positive	Unstable node	Trajectories diverge from (0, 0)
Real, one positive, one negative	Saddle point	Trajectories approach along some directions and diverge along others
Complex with negative real part	Stable focus / spiral	Trajectories spiral into (0, 0)
Complex with positive real part	Unstable focus / spiral	Trajectories spiral away from (0, 0)
Purely imaginary eigenvalues	Center	Trajectories form closed orbits

around (0, 0) |

Calculating Eigenvalues

For the Jacobian matrix:

$$
J = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
$$

the eigenvalues λ satisfy:

$$
\lambda^2 - \text{trace}(J)\lambda + \det(J) = 0
$$

with:

- $\text{trace}(J) = a_{11} + a_{22}$
- $\det(J) = a_{11}a_{22} - a_{12}a_{21}$

Example:
Using the earlier Jacobian:

$$
J = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
$$

- $\text{trace} = 1 + 1 = 2$
- $\det = (1)(1) - (-1)(1) = 1 + 1 = 2$

Eigenvalues:

$$
\lambda^2 - 2\lambda + 2 = 0
$$

Discriminant:

$$
D = 4 - 8 = -4 < 0
$$

Eigenvalues are complex conjugates:

$$
\lambda = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i
$$

Since the real part is positive, the critical point is an unstable focus with spiral trajectories moving away from (0, 0).

Conclusion and Summary

Analyzing the critical point at (0, 0) involves a systematic approach:

1. Verification: Confirm that (0, 0) is an equilibrium by evaluating the system functions at that point.
2. Local Linearity: Use the Jacobian matrix to linearize the system around (0, 0), providing a simplified local model.
3. Classification: Determine the nature of the equilibrium by calculating eigenvalues of the Jacobian, leading to insights about stability and

Frequently Asked Questions

How can I verify if (0,0) is a critical point for a given system?

To verify if (0,0) is a critical point, compute the system's derivatives at (0,0). If both derivatives are zero at that point, then (0,0) is a critical point.

What does it mean for a system to be locally linear around a point?

A system is locally linear around a point if it can be approximated by its tangent (linear) approximation at that point, typically via the Jacobian matrix evaluated there.

How do I show that a system is locally linear near (0,0)?

Calculate the Jacobian matrix of the system at (0,0). If the system can be expressed as the Jacobian times the variables plus higher-order terms that are negligible near (0,0), then the system is locally linear there.

What is the significance of analyzing the type of critical point at (0,0)?

Determining the type (e.g., node, saddle, focus) helps understand the local behavior of the system, such as stability and the nature of trajectories near the critical point.

How do I classify the critical point at (0,0) once I have the Jacobian matrix?

Calculate the eigenvalues of the Jacobian matrix at (0,0). The sign and nature of these eigenvalues determine whether the critical point is stable, unstable, a saddle, focus, or node.

What are the steps to verify that (0,0) is a critical point and analyze its stability?

First, set the system equations to zero and solve for (0,0). Next, compute the Jacobian at (0,0), find its eigenvalues, and interpret these to classify the critical point's stability and type.

Can a critical point at (0,0) be non-hyperbolic, and what does that imply?

Yes, if the Jacobian has eigenvalues with zero real parts, the critical point is non-hyperbolic. This typically requires further analysis (e.g., center manifold theory) to determine stability and behavior.

Why is it important to show that the system is locally linear when analyzing (0,0)?

Because the local linear approximation simplifies the analysis of stability and type of the critical point, making it easier to classify behavior near (0,0).

What tools or methods are commonly used to analyze the type of a critical point at (0,0)?

Common methods include computing the Jacobian matrix, finding its eigenvalues, performing phase plane analysis, and using stability criteria like the Hartman-Grobman theorem.

If the Jacobian at (0,0) has real positive eigenvalues, what is the type of the critical point?

The critical point is an unstable node because trajectories move away from

(0,0) along the eigenvectors corresponding to positive eigenvalues.

Additional Resources

Verify That (0, 0) Is A Critical Point, Show That The System Is Locally Linear, And Discuss The Type

Understanding the nature and classification of equilibrium points in dynamical systems is fundamental in the qualitative analysis of differential equations. Specifically, verifying whether a particular point, such as (0, 0), is a critical point, demonstrating the local linearity of the system near that point, and determining its type (e.g., node, saddle, focus, or center) are essential steps. This comprehensive review aims to elucidate these concepts, providing detailed insights into each aspect.

Verifying That (0, 0) Is a Critical Point

Definition of Critical (Equilibrium) Points

In a dynamical system described by a set of differential equations:

$$
\begin{cases}
\frac{dx}{dt} = f(x, y), \\
\frac{dy}{dt} = g(x, y),
\end{cases}
$$

a point (x_0, y_0) is called a critical point or equilibrium point if the system is at rest there, meaning:

$$
f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.
$$

This indicates that at (x_0, y_0), the system’s trajectories are neither moving away nor approaching, but are momentarily static.

Verifying (0, 0) as a Critical Point

To verify whether $(0, 0)$ is a critical point, follow these steps:

1. Identify the functions $f(x, y)$ and $g(x, y)$:

- These functions typically come from the differential equations defining the system.

2. Substitute $(0, 0)$ into $f(x, y)$ and $g(x, y)$:

- Calculate $f(0, 0)$ and $g(0, 0)$.

3. Confirm that both are zero:

- If $f(0, 0) = 0$ and $g(0, 0) = 0$, then $(0, 0)$ is a critical point.

Example:

Suppose the system is:

$$
\begin{cases}
\frac{dx}{dt} = x - y, \\
\frac{dy}{dt} = x + y.
\end{cases}
$$

Evaluate at $(0, 0)$:

$$
f(0, 0) = 0 - 0 = 0, \quad g(0, 0) = 0 + 0 = 0.
$$

Thus, $(0, 0)$ is a critical point.

4. General verification:

- For an arbitrary system, always verify explicitly by substitution.

Showing That The System Is Locally Linear Near the Critical Point

What Is Local Linearity?

A dynamical system is said to be locally linear near a critical point if, in some neighborhood around that point, the system behaves approximately like a linear system. This concept is crucial because linear systems are well-understood, and their qualitative behavior can be classified comprehensively.

Mathematically:

Near (x_0, y_0), the system can be expressed as:

$$
\begin{cases}
\frac{dx}{dt} \approx A_{11}(x - x_0) + A_{12}(y - y_0), \\
\frac{dy}{dt} \approx A_{21}(x - x_0) + A_{22}(y - y_0),
\end{cases}
$$

where $A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}$ is the Jacobian matrix evaluated at the critical point.

Constructing the Jacobian Matrix

Step-by-step:

1. Calculate partial derivatives:

$$
A = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
$$

2. Evaluate at the critical point:

$$
A|_{(x_0, y_0)} = \left. \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix} \right|_{(x_0, y_0)}
$$

3. Approximate the system:

- Near (x_0, y_0), the nonlinear system behaves like:

$$
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} \approx A \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}
$$

Example:

Using the previous system:

$$
f(x, y) = x - y, \quad g(x, y) = x + y,
$$

the Jacobian is:

$$
A = \begin{bmatrix}
\frac{\partial}{\partial x}(x - y) & \frac{\partial}{\partial y}(x - y) \\
\frac{\partial}{\partial x}(x + y) & \frac{\partial}{\partial y}(x + y)
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}.
$$

Evaluating at $(0, 0)$ (which is the same point), the Jacobian remains:

$$
A = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}.
$$

4. Justification of local linearity:

- The nonlinear system can be approximated by this linear system in a small neighborhood around $(0, 0)$.

- This approximation is valid because higher-order terms become negligible as the neighborhood shrinks, ensuring that the linear system captures the essence of the system’s behavior near the equilibrium.

Implications of Local Linearity

- The linear approximation simplifies the analysis of stability and qualitative behavior.

- The eigenvalues of the Jacobian matrix determine the local dynamics.

- If eigenvalues have negative real parts, the equilibrium is locally asymptotically stable; if positive, unstable; and if purely imaginary, the system exhibits center behavior, often necessitating nonlinear analysis.

Classifying the Type of Critical Point

Eigenvalues and Their Role

The eigenvalues of the Jacobian matrix A at the critical point are central to classifying its nature:

- Real, distinct eigenvalues:

- Both negative: Stable node

- Both positive: Unstable node

- Opposite signs: Saddle point

- Complex conjugate eigenvalues:

- Negative real part: Stable focus or spiral

- Positive real part: Unstable focus or spiral

- Zero real part: Center (requires further nonlinear analysis)

Calculating Eigenvalues

Given the Jacobian matrix A, the eigenvalues λ satisfy:

$$
\det(A - \lambda I) = 0,
$$

which simplifies to:

$$
\lambda^2 - \text{tr}(A) \lambda + \det(A) = 0,
$$

where:

- $\text{tr}(A) = A_{11} + A_{22}$,

- $\det(A) = A_{11}A_{22} - A_{12}A_{21}$.

The roots are:

$$
\lambda_{1,2} = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4 \det(A)}}{2}.
$$

Classification Criteria Based on Eigenvalues

Eigenvalues	System Behavior	Critical Point Type	Stability
Real, same sign, distinct	Both negative	Node (attracting)	Stable
Real, same sign, distinct	Both positive	Node (repelling)	Unstable
Real, opposite signs	One positive, one negative	Saddle	Unstable
Complex conjugates with negative real part	-	Focus/Spiral	Stable (sink)
Complex conjugates with positive real part	-	Focus/Spiral	Unstable (source)
Purely imaginary eigenvalues	-	Center	Neutral (linear analysis inconclusive)

Examples of Classification

- Saddle Point:

For the earlier example:

$$
A = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
$$

Verify That 0 0 Is A Critical Point Show That The System Is Locally Linear And Discuss The Type

Verify That 0 0 Is A Critical Point Show That The System Is Locally Linear And Discuss The Type

Related Articles

- What Is The Difference Between Independent And Dependent Demand. Group Of Answer Choices Independent
- The Random Variable Y = Ln(X) Has A Normal Distribution With Mean Of 10 And Standard Deviation Of 2,
- True Or False? The Average Level Of Tariffs On Imported Products Charged By Industrialized Countries

Back to Home