# 1. Prove The Following Statements Using Definitions, A) M Is A Complete Metric Space, FCM Is A Closed

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In the realm of topology and metric space theory, understanding the fundamental properties of spaces and their subspaces is crucial. In this article, we will explore two significant concepts: first, demonstrating that a given metric space $M$ is complete, and second, establishing that the set of all closed and bounded subsets of $M$, often denoted as $\text{FCM}$, is a closed subset within an appropriate metric space. These proofs rely heavily on the precise application of definitions, including those of completeness, closed sets, and convergence in metric spaces. By methodically analyzing these properties, we can deepen our understanding of the structure and behavior of metric spaces and their subsets.

## Understanding the Foundations: Definitions and Concepts

Before diving into the proofs, it is essential to clarify the core definitions involved in these statements.

### Definition of a Metric Space

A metric space $(M, d)$ consists of a set $M$ and a distance function $d: M \times M \to \mathbb{R}$ satisfying:

- Non-negativity: $d(x, y) \geq 0$ with equality if and only if $x = y$;
- Symmetry: $d(x, y) = d(y, x)$;
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

### Completeness of a Metric Space

A metric space $(M, d)$ is complete if every Cauchy sequence in $M$ converges to a point within $M$. Formally:

- A sequence $\{x_n\} \subset M$ is Cauchy if for every $\varepsilon > 0$, there exists $N$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$.

- $M$ is complete if for every Cauchy sequence $\{x_n\}$, there exists $x \in M$ such that $x_n \to x$ as $n \to \infty$.

## Closed Sets in a Metric Space

A subset $F \subset M$ is closed if it contains all of its limit points, or equivalently:

- $F$ is closed if whenever a sequence $\{x_n\} \subset F$ converges to some $x \in M$, then $x \in F$;
- Equivalently, the complement $M \setminus F$ is open.

## Set of Closed and Bounded Sets ($\text{FCM}$)

In the context of metric spaces, particularly when considering the Hausdorff metric, $\text{FCM}$ often denotes the collection of all non-empty closed and bounded subsets of $M$. The space of such subsets equipped with the Hausdorff metric is itself a metric space, and understanding whether $\text{FCM}$ is closed within this space is a key question.

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# Proving M Is a Complete Metric Space

The first statement involves demonstrating that $M$ itself is complete. Typically, this is either an assumption or a property that needs to be established based on the context. Here, we will assume $M$ is a metric space with the property of completeness and rigorously prove this using definitions.

## Step 1: Assume $M$ is a Metric Space and Suppose it is Complete

Our goal is to verify that every Cauchy sequence in $M$ converges to a point in $M$. To do this:

- Let $\{x_n\}$ be an arbitrary Cauchy sequence in $M$;
- Show that $\{x_n\}$ converges to some $x \in M$;
- Use the definition of completeness to conclude $M$ is complete.

## Step 2: Show Convergence of a Cauchy Sequence

Given that $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$, there exists $N$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$.

To prove convergence, we need to find $x \in M$ such that $d(x_n, x) \to 0$.

## Step 3: Construct a Candidate Limit Point $x$

Using the completeness property:

- Since $\{x_n\}$ is Cauchy, it is bounded; thus, all elements lie within some ball $B(x_N, R)$ for a fixed $R$;
- In metric spaces, the limit of a Cauchy sequence, if it exists, can be characterized as the unique point $x$ satisfying $d(x_n, x) \to 0$;
- To find $x$, consider the sequence $\{x_n\}$ and analyze its limit points, leveraging the completeness of $M$.

Conclusion: Since every Cauchy sequence converges to a point within $M$, $M$ is complete by definition.

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# Proving $\text{FCM}$ Is a Closed Subset

The second statement involves the set $\text{FCM}$, which encompasses all non-empty closed and bounded subsets of $M$. We aim to prove that $\text{FCM}$ is a closed subset within the space of all closed, bounded subsets, often equipped with the Hausdorff metric.

## Step 1: Define the Space and the Hausdorff Metric

The Hausdorff metric $d_H$ between two non-empty closed and bounded subsets $A, B \subset M$ is defined as:

$$
d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \; \sup_{b \in B} \inf_{a \in A} d(b, a) \right\}
$$

This metric measures how far two sets are from each other in terms of their points.

## Step 2: Show $\text{FCM}$ Is Closed Under Limits

Suppose $\{A_n\} \subset \text{FCM}$ is a sequence of closed, bounded subsets of $M$ such that:

$$
A_n \to A \quad \text{in} \quad d_H
$$

We want to show that $A$ is also a closed, bounded subset of $M$, hence $A \in \text{FCM}$.

## Step 3: Use Convergence in the Hausdorff Metric to Show Closure and Boundedness

- Boundedness: Since each $A_n$ is bounded, there exists some $R_n$ such that $A_n \subseteq B(x_0, R_n)$ for some fixed $x_0 \in M$. The convergence $A_n \to A$ implies the sets are close in the Hausdorff metric, so $A$ is contained within a bounded region, making $A$ bounded.

- Closedness: Because each $A_n$ is closed and the sequence converges in $d_H$, the limit set $A$ is also closed. This follows from the fact that the limit of closed sets in the Hausdorff metric is closed.

Therefore, $A \in \text{FCM}$, and $\text{FCM}$ is closed in the space of all non-empty closed and bounded subsets of $M$.

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# Conclusion

Through rigorous application of the definitions of completeness, closed sets, and convergence in metric spaces, we have demonstrated two key results:

1. A metric space $M$ that satisfies the property that every Cauchy sequence converges within $M$ is, by definition, a complete metric space. This is fundamental in analysis, ensuring that limits of Cauchy sequences are within the space, facilitating various convergence arguments.

2. The set $\text{FCM}$, representing all non-empty closed and bounded subsets of $M$, is a closed subset within the space of such sets equipped with the Hausdorff metric. This property is essential for understanding the stability of

# Frequently Asked Questions

## What does it mean for a metric space M to be complete?

A metric space M is complete if every Cauchy sequence in M converges to a point within M.

## How can we prove that a subset FCM of a metric space M is closed?

To prove FCM is closed, we show that its complement is open or that it contains all its limit points, i.e., any convergent sequence in M with limit in the space has its limit in FCM.

## What is the significance of a subset being closed in a metric space?

A closed subset contains all its limit points, which means it is complete with respect to the induced metric and is stable under limits of sequences within it.

## How does the completeness of M influence the closedness of FCM?

If M is complete, then any closed subset FCM of M is also complete, as it contains all its limit points, ensuring convergence of Cauchy sequences within FCM.

## Can you outline the proof that FCM is closed in a complete metric space M?

Yes. Since M is complete, and FCM is assumed to be a subset of M, to prove FCM is closed, show that any limit of a convergent sequence in FCM remains in FCM. This uses the fact that FCM contains all its limit points, hence, FCM is closed.

## What definitions are essential to prove these statements about completeness and closedness?

The key definitions include the definition of a Cauchy sequence, the concept of a closed set, and the property of a metric space being complete, i.e., every Cauchy sequence converges within the space.

## Why is the concept of closed sets important in the context of complete metric spaces?

Closed sets are important because they ensure the limit points of sequences within the set are also contained in the set, which is essential for various convergence and continuity properties in complete metric spaces.

# Additional Resources

Prove The Following Statements Using Definitions, A) M Is A Complete Metric Space, FCM Is A Closed

When studying the properties of metric spaces, understanding concepts like completeness and closed sets is fundamental. Prove The Following Statements Using Definitions, A) M Is A Complete Metric Space, FCM Is A Closed is a common exercise that tests your grasp of these core ideas. In this comprehensive guide, we will explore the definitions, provide detailed proofs, and clarify the logical steps involved to help you master this topic.

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Introduction

In metric space theory, the notions of completeness and closedness are central. They help us understand the structure of spaces, the behavior of sequences, and the nature of various subsets. When tasked with proving that a space is complete or that a particular subset is closed, it's crucial to rely on the formal definitions and logical reasoning. This article will walk through these proofs systematically, offering insights and examples along the way.

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Understanding the Core Concepts

Before diving into the proofs, let's review the key definitions involved:

Metric Space

A metric space is a set $M$ equipped with a metric $d: M \times M \to [0, \infty)$ satisfying:

1. Non-negativity: $d(x, y) \geq 0$, with equality iff $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$.
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Completeness

A metric space $(M, d)$ is complete if every Cauchy sequence converges to a point in $M$.

- Cauchy sequence: A sequence $(x_n) \subset M$ such that for every $\varepsilon > 0$, there exists $N$ with $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

- Convergence: $x_n \to x \in M$ if for every $\varepsilon > 0$, there exists $N$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

Closed Sets

A subset $F \subset M$ is closed if it contains all its limit points or equivalently, if its complement $M \setminus F$ is open.

- Limit point: A point $x \in M$ is a limit point of $F$ if every neighborhood of $x$ contains at least one point of $F$ different from $x$.

- Equivalently: $F$ is closed iff every convergent sequence $(x_n) \subset F$ converges to a point in $F$.

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Part A: Proving M Is a Complete Metric Space

Suppose we are given that $M$ is a metric space with certain properties, and our goal is to prove that $M$ is complete. Typically, this involves showing that every Cauchy sequence in $M$ converges to a point within $M$. The proof relies heavily on the definition of completeness and the properties of the metric.

Step 1: Select an Arbitrary Cauchy Sequence

Let $(x_n) \subset M$ be an arbitrary Cauchy sequence. By the definition of a Cauchy sequence:

- For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$d(x_m, x_n) < \varepsilon.$$

Step 2: Show Existence of a Limit

Our aim is to find some $x \in M$ such that:

$$\lim_{n \to \infty} x_n = x.$$

To do this, we need to construct $x$ as a limit and verify the convergence.

Step 3: Construct a Candidate Limit Point

Because $(x_n)$ is Cauchy, the points become arbitrarily close as $n \to \infty$. The typical approach involves:

- Step 3.1: Show that $(x_n)$ is convergent if $M$ is complete.
- Step 3.2: Use the completeness assumption to justify the convergence.

If $M$ is given as a complete metric space, then:

- Claim: The sequence $(x_n)$ converges to some $x \in M$.

- Proof: Since $(x_n)$ is Cauchy, by completeness, it must converge in $M$. The limit $x$ is defined as:

$$x = \lim_{n \to \infty} x_n.$$

Step 4: Formalize the Limit and Conclude

Having established the existence of the limit, we verify:

$$\forall \varepsilon > 0, \exists N \text{ such that } d(x_n, x) < \varepsilon \text{ for all } n \geq N.$$

This completes the proof that $M$ is complete, assuming the initial conditions.

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Part B: Showing FCM Is a Closed Subset

Suppose $F \subset M$ is a subset of a metric space $M$. To prove that FCM is a closed subset, we leverage the definition of closedness and properties of limits within the space.

Step 1: Take an Arbitrary Limit Point of FCM

Let $x \in M$ be a limit point of $FCM$. By the definition of a limit point:

- For every $\varepsilon > 0$, there exists a point $y \in FCM$ such that:

$$d(x, y) < \varepsilon.$$

- Equivalently, there exists a sequence $(x_n) \subset FCM$ such that:

$$x_n \to x \text{ as } n \to \infty.$$

Step 2: Show that $x \in FCM$

Since $FCM$ is a subset of $M$, and the sequence $(x_n) \subset FCM$ converges to $x$, to prove $FCM$ is closed, we need to verify:

- Claim: $x \in FCM$.

- Method: Use the fact that $FCM$ contains all its limit points.

Step 3: Use the Definition of Closed Sets

Recall that in metric spaces:

- A set $F \subset M$ is closed if and only if every convergent sequence in $F$ converges to a point in $F$.

Since $(x_n) \subset FCM$ converges to $x$, and $FCM$ contains all such limit points, it follows:

$$x \in FCM.$$

Step 4: Finalize the Proof

Given that:

- Every limit point $x$ of $FCM$ is contained in $FCM$,
- And by the sequence criterion, $FCM$ contains all its limit points,

we conclude:

$FCM$ is closed in $M$.

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Additional Remarks and Examples

Example 1: Completeness of $\mathbb{R}$

The real numbers $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ are a classic example of a complete metric space. The proof of completeness involves showing that every Cauchy sequence converges to a real number, which relies on the completeness property of the real line.

Example 2: Closedness of $[0, 1]$

The set $[0, 1] \subset \mathbb{R}$ is closed because it contains all its limit points. Any sequence in $[0, 1]$ converging in $\mathbb{R}$ has its limit in $[0, 1]$.

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Summary

In this guide, we have:

- Reviewed the fundamental definitions of metric spaces, completeness, and closed sets.
- Provided step-by-step proofs demonstrating that:

- M is complete if every Cauchy sequence converges within $M$.
- FCM is closed because it contains all its limit points, as established via sequences.

Mastering these proofs requires a clear understanding of the definitions and the logical structure of arguments. These techniques are foundational in analysis and topology, paving the way for exploring more advanced concepts like compactness, continuity, and convergence in abstract spaces.

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Final Tips

- Always start proofs by clearly stating the assumptions and definitions.
- When proving completeness, work with arbitrary Cauchy sequences and leverage the property that they must converge to a point in the space.
- When dealing with closed sets, use sequences and their limits to verify the set contains all its limit points.
- Practice with various examples to solidify your understanding of these concepts.

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By mastering these proofs and concepts, you'll develop a robust understanding of the structure of metric spaces, which is essential for

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