11. Use Taylor's Formula To Find The First Four Nonzero Terms Of The Taylor Series Expansion For F(1)

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In calculus, Taylor series play a fundamental role in approximating functions near a specific point. When dealing with a function $f(x)$, the Taylor series expansion around a point a provides a polynomial expression that approximates the function's behavior near that point. This technique is especially useful when direct evaluation of the function is complex or impossible, but derivatives are accessible. In this article, we will explore how to utilize Taylor's formula to find the first four nonzero terms of the Taylor series expansion centered at $a=1$ for a given function $f(x)$. The process involves understanding Taylor's theorem, calculating derivatives at the point $a=1$, and assembling the series accordingly.

Understanding Taylor's Formula

Definition of Taylor Series

The Taylor series of a function $f(x)$ centered at a is given by:

$$
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
$$

where:

- $f^{(n)}(a)$ is the n-th derivative of f evaluated at a.
- $n!$ is the factorial of n.

Taylor's Theorem

Taylor's theorem states that, for a sufficiently smooth function f, the Taylor series centered at a converges to $f(x)$ within a certain radius of convergence. The partial sum of the first n terms provides an approximation:

$$
f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
$$

The accuracy improves as n increases, especially near a. For our purpose, we are interested in the first four nonzero terms of this expansion at $a=1$.

Step-by-Step Process to Find the Expansion

Step 1: Identify the Function $f(x)$

To proceed, specify the function $f(x)$. For illustration, suppose we are given:

$$
f(x) = \frac{1}{(1 - x)^k}
$$

where k is a positive integer. Alternatively, the process applies generally; if you have a specific function, the method remains the same.

Step 2: Compute Derivatives at $x=1$

Calculate the derivatives $f^{(n)}(x)$ and evaluate them at $a=1$. This requires applying differentiation rules suitable for the function. For example, if $f(x) = (1 - x)^{-k}$, derivatives involve the generalized binomial theorem or repeated application of chain rule.

Step 3: Determine the First Several Derivatives

Compute derivatives systematically:

1. Find $f(1)$.

2. Find $f'(x)$ and evaluate at $x=1$.

3. Find $f''(x)$ and evaluate at $x=1$.

4. Find $f'''(x)$ and evaluate at $x=1$.

5. Find $f^{(4)}(x)$ and evaluate at $x=1$.

This sequence provides the necessary coefficients for the Taylor polynomial.

Step 4: Construct the Taylor Series Terms

Using the derivatives, write the partial sum up to the order needed to identify the first four nonzero terms:

$$
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x - 1)^k
$$

Identify which derivatives are nonzero and include only those terms until four terms are obtained.

Step 5: Simplify and Identify the Nonzero Terms

Combine the terms, simplify, and verify which are nonzero to ensure the accuracy of the first four nonzero terms. These terms form the initial approximation of $f(x)$ near $x=1$.

Illustrative Example

Example Function: $f(x) = \frac{1}{(1 - x)^2}$

Let's demonstrate the process with this specific function to clarify the steps involved.

Step 1: Compute $f(1)$

$$
f(1) = \frac{1}{(1 - 1)^2} = \frac{1}{0}
$$

which is undefined, indicating a singularity at $x=1$. However, we can analyze the function's behavior around $a=1$ by considering the Laurent series or noting that the derivatives may still provide finite coefficients for a series expansion in powers of $(x-1)$. Alternatively, choose a different function with a finite value at $a=1$, such as $f(x) = e^{x}$, for simplicity.

Alternative Example: $f(x) = e^{x}$ centered at $a=1$.

Step 1: Derivatives of $f(x) = e^{x}$

$$
f^{(n)}(x) = e^{x}
$$

Thus, at $x=1$:

$$
f^{(n)}(1) = e
$$

Step 2: Write the Taylor series expansion

$$
f(x) = e^{x} = \sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n
$$

since all derivatives are e at $x=1$.

Step 3: Find the first four nonzero terms

$$
f(x) = e + e (x - 1) + \frac{e}{2!} (x - 1)^2 + \frac{e}{3!} (x - 1)^3 + \cdots
$$

The first four nonzero terms are:

$$
\boxed{
e + e (x - 1) + \frac{e}{2} (x - 1)^2 + \frac{e}{6} (x - 1)^3
}
$$

Note: If the function has a different form, the derivatives and resulting series will differ. The essential steps remain consistent:

- Compute derivatives at $a=1$.
- Use the derivatives to write terms of the Taylor expansion.
- Identify the first four nonzero terms.

Special Considerations and Common Challenges

Handling Functions with Singularities at $a=1$

Some functions may not be defined at the expansion point $a=1$, such as in the earlier example $f(x) = \frac{1}{(1 - x)^2}$. In such cases, the Taylor series may be replaced by a Laurent series, which includes negative powers of $(x - 1)$. Alternatively, one could consider the expansion around a different point where the function is analytic.

Ensuring Derivatives Are Computable

For functions where derivatives are complex or not straightforward, consider using known series expansions, binomial series, or other approximation techniques to identify the pattern of derivatives.

Determining Nonzero Terms

Some derivatives may evaluate to zero at $a=1$. Only the derivatives that produce nonzero coefficients will contribute to the nonzero terms of the Taylor series. It is critical to compute derivatives carefully and verify their values at the center point.

Conclusion

Using Taylor's formula to find the first four nonzero terms of a Taylor series expansion involves a systematic approach: selecting the function, computing derivatives at the point of expansion, plugging these derivatives into the Taylor series formula, and simplifying to identify the initial nonzero terms. This process not only provides an approximation for the function near the point $a=1$ but also deepens understanding of the function's local behavior. Mastery of differentiation and series expansion techniques is essential for applying Taylor's theorem effectively across various functions and contexts.

Frequently Asked Questions

What is Taylor's Formula and how is it used to find the Taylor series expansion of a function at a point?

Taylor's Formula provides an approximation of a function near a point by expressing it as an infinite sum of derivatives at that point. It is used to find the Taylor series expansion by calculating derivatives of the function at a specific point and combining them with factorial terms to form the series.

How do you determine the derivatives needed for the Taylor series expansion of F(x) at x=1?

To determine the derivatives, you differentiate the function F(x) repeatedly, evaluate each derivative at x=1, and then use these values in the Taylor series formula to find the expansion terms.

What is the general form of the Taylor series expansion for a function F(x) centered at x=1?

The general form is F(1) + F'(1)(x-1) + (F''(1)/2!)(x-1)^2 + (F'''(1)/3!)(x-1)^3 + ... , where each derivative is evaluated at x=1.

How do you identify the first four nonzero terms of the Taylor series for a function at x=1?

You compute derivatives of the function at x=1, identify the first four derivatives that are nonzero, and then write the corresponding terms from the Taylor series formula, including factorial denominators.

Why is it important to find the first four nonzero terms of the Taylor series expansion?

Finding these terms helps approximate the function accurately near the point x=1, especially when the series converges quickly, and provides insight into the function's behavior around that point.

Can you give an example of applying Taylor's Formula to find the first four nonzero terms of F(x) at x=1?

Yes. For example, for F(x)=e^x, at x=1, the derivatives are all e^x, so F(1)=e, F'(1)=e, F''(1)=e, F'''(1)=e. The first four nonzero terms are e + e(x-1) + (e/2!)(x-1)^2 + (e/3!)(x-1)^3.

What challenges might arise when using Taylor's Formula to find the series expansion at x=1?

Challenges include computing higher-order derivatives accurately, especially for complex functions, and determining where derivatives become zero, which affects the number of nonzero terms.

How does the convergence of the Taylor series impact the usefulness of the first four terms in approximating F(x)?

If the series converges rapidly around x=1, the first four terms provide a good approximation. If convergence is slow, more terms may be needed for an accurate approximation.

What are some common functions where Taylor's Series expansion around x=1 is particularly useful?

Functions like exponential functions, logarithms, and trigonometric functions are common candidates, especially when analyzing behavior near x=1 or simplifying calculations involving these functions.

Additional Resources

Use Taylor's Formula To Find The First Four Nonzero Terms Of The Taylor Series Expansion For F(1)

Understanding how to utilize Taylor's formula to find the first few nonzero terms of a Taylor series expansion is an essential skill in calculus, especially in approximation techniques and analyzing the behavior of functions near specific points. This process allows mathematicians and students to approximate complicated functions with polynomials, which are easier to work with and analyze. In this article, we will explore the step-by-step methodology of applying Taylor's formula to find the first four nonzero terms of the Taylor series expansion for a function evaluated at a specific point, specifically for $F(1)$. We will break down the theoretical foundations, detailed procedures, and practical considerations involved in this process.

Understanding Taylor's Formula

Before we delve into the application, it is crucial to grasp what Taylor's formula entails. Taylor's theorem provides a way to approximate a sufficiently smooth function $f(x)$ around a point a by a polynomial whose coefficients involve derivatives of the function at a.

Taylor's Formula (General Form):

$$
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)
$$

where:

- $f^{(n)}(a)$ denotes the nth derivative of f evaluated at a,
- $R_n(x)$ is the remainder term, which tends to zero as $n \to \infty$ under certain conditions.

Features and Significance:

- Provides polynomial approximations with increasing accuracy as more terms are added.
- Facilitates local analysis of functions near the point a.
- Essential in numerical methods and theoretical analysis.

Choosing the Function and Point of Expansion

In our specific case, the task is to find the first four nonzero terms of the Taylor series expansion for $F(x)$ at $x=1$. To proceed, we need:

- The explicit form of the function $F(x)$.
- The value of the derivatives at $x=1$.

Suppose, for illustration purposes, we consider a function such as:

$$
F(x) = e^{x} \sin x
$$

and we want to find the Taylor expansion around $a=1$.

Note: The actual function $F(x)$ should be specified in your problem statement; if not, choose a function for demonstration.

Step-by-Step Process to Find the First Four Nonzero Terms

Step 1: Compute $F(1)$

Evaluate the function at $x=1$:

$$
F(1) = e^{1} \sin 1
$$

Calculate the numerical value (if needed):

$$
F(1) \approx e \times \sin 1 \approx 2.7183 \times 0.8415 \approx 2.2909
$$

Step 2: Find Derivatives of $F(x)$

Calculate the derivatives $F'(x)$, $F''(x)$, $F'''(x)$, etc., and evaluate at $x=1$.

For $F(x) = e^{x} \sin x$:

- First derivative:

$$
F'(x) = \frac{d}{dx}(e^{x} \sin x) = e^{x} \sin x + e^{x} \cos x = e^{x} (\sin x + \cos x)
$$

- Second derivative:

$$
F''(x) = \frac{d}{dx}[e^{x} (\sin x + \cos x)] = e^{x} (\sin x + \cos x) + e^{x} (\cos x - \sin x) = e^{x} [(\sin x + \cos x) + (\cos x - \sin x)] = e^{x} (2 \cos x)
$$

- Third derivative:

$$
F'''(x) = \frac{d}{dx} [e^{x} (2 \cos x)] = e^{x} (2 \cos x) + e^{x} (-2 \sin x) = e^{x} [2 \cos x - 2 \sin x]
$$

Evaluate these derivatives at $x=1$:

$$
F(1) \approx 2.2909
$$
$$
F'(1) = e^{1} (\sin 1 + \cos 1) \approx 2.7183 (0.8415 + 0.5403) \approx 2.7183 \times 1.3818 \approx 3.757
$$

$$
F''(1) = e^{1} (2 \cos 1) \approx 2.7183 \times 2 \times 0.5403 \approx 2.7183 \times 1.0806 \approx 2.941
$$

$$
F'''(1) = e^{1} [2 \cos 1 - 2 \sin 1] \approx 2.7183 [2 \times 0.5403 - 2 \times 0.8415] = 2.7183 (1.0806 - 1.683) \approx 2.7183 \times (-0.6024) \approx -1.636
$$

Step 3: Write the Taylor Series Terms

Using the derivatives, write the series expansion around $a=1$:

$$
F(x) \approx F(1) + F'(1)(x-1) + \frac{F''(1)}{2!}(x-1)^2 + \frac{F'''(1)}{3!}(x-1)^3 + \cdots
$$

Plugging in the values:

$$
F(x) \approx 2.2909 + 3.757 (x-1) + \frac{2.941}{2} (x-1)^2 + \frac{-1.636}{6} (x-1)^3
$$

Simplify coefficients:

$$
F(x) \approx 2.2909 + 3.757 (x-1) + 1.4705 (x-1)^2 - 0.273 (x-1)^3
$$

Step 4: Identify the First Four Nonzero Terms

The first four nonzero terms are:

1. $F(1) \approx 2.2909$

2. $F'(1) (x - 1) \approx 3.757 (x-1)$

3. $\frac{F''(1)}{2!} (x-1)^2 \approx 1.4705 (x-1)^2$

4. $\frac{F'''(1)}{3!} (x-1)^3 \approx -0.273 (x-1)^3$

These terms collectively provide a polynomial approximation of $F(x)$ near $x=1$.

Practical Considerations & Common Challenges

Applying Taylor's formula in practice involves several considerations:

Pros:

- High accuracy near the expansion point: Taylor series provide excellent local approximations.
- Systematic approach: Derivative calculations follow a clear pattern, especially with the use of rules like product rule, chain rule, etc.
- Flexible: Can be applied to virtually any infinitely differentiable function.

Cons:

- Limited radius of convergence: Taylor series may not approximate well far from the point a.
- Computational complexity: Derivatives of complex functions can be tedious to compute.
- Zero derivatives: If derivatives at a are zero, higher-order terms might be needed to find nonzero contributions, potentially delaying the identification of the first four nonzero terms.

Tips for Effective Application:

- Use symbolic computation tools (e.g., WolframAlpha, Mathematica) for derivatives of complicated functions.
- Always verify the derivatives to avoid errors.
- Be aware of the function's behavior to determine the radius of convergence.

Summary and Final Thoughts

In conclusion, using Taylor's formula to find the first four nonzero terms of the Taylor series expansion for $F(1)$ involves:

- Selecting the point of expansion $a=1$.
- Computing derivatives of the function at $a=1$.
- Substituting these derivatives into the Taylor series formula.
- Simplifying to identify the initial nonzero polynomial terms.

This process not only provides a powerful approximation technique but also deepens the understanding of the local behavior of functions. Whether in pure mathematics, physics, or engineering, mastering the application of Taylor's formula is invaluable for analyzing complex functions and developing approximations with practical significance.

Features Recap:

- Precise local approximation of functions.
- Foundation for numerical methods.
- Useful in solving differential equations, optimization, and modeling.

Limitations Recap:

- Only valid near the expansion point.
- Calculating derivatives can be complex for some functions.
- Convergence issues for certain functions.

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