2. (10 Points) Find The 4-point Discrete Fourier Transform (DFT) Of The Sequence X(n) = {1, 3, 3, 4}.

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Understanding the Discrete Fourier Transform (DFT) is fundamental in digital signal processing, communications, and many engineering applications. This article provides a comprehensive, step-by-step explanation of how to compute the 4-point DFT of the sequence X(n) = {1, 3, 3, 4}. We will explore the theoretical foundations, detailed calculations, and practical implications, making this guide valuable for students, engineers, and anyone interested in Fourier analysis.

Introduction to Discrete Fourier Transform (DFT)

The DFT transforms a finite sequence of complex numbers from the time domain into the frequency domain. It reveals the frequency components present in a discrete signal, enabling analysis, filtering, and spectral estimation.

Definition of the 4-point DFT

For a sequence $x(n)$ of length $N=4$, the DFT is defined as:

$$
X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j 2 \pi kn / N}
$$

where:
- $x(n)$ is the input sequence,
- $X(k)$ is the DFT output,
- $k = 0, 1, 2, 3$,
- j is the imaginary unit.

This transformation converts the sequence from its original domain into a set of frequency components, each corresponding to a specific frequency bin.

Step-by-Step Calculation of the 4-point DFT for X(n) = {1, 3, 3, 4}

Let's proceed to compute $X(k)$ for $k = 0, 1, 2, 3$. The process involves substituting the sequence values into the DFT formula and evaluating the sums.

Given Sequence

$$
x(0) = 1, \quad x(1) = 3, \quad x(2) = 3, \quad x(3) = 4
$$

Computing $X(0)$

$$
X(0) = \sum_{n=0}^{3} x(n) \cdot e^{-j 2 \pi \cdot 0 \cdot n / 4} = \sum_{n=0}^{3} x(n) \cdot 1
$$

Since $e^{0} = 1$:

$$
X(0) = 1 + 3 + 3 + 4 = 11
$$

This component represents the total sum of the sequence, often called the DC component.

Computing $X(1)$

$$
X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-j 2 \pi \cdot 1 \cdot n / 4}
$$

The exponential terms are:

- For $n=0$:

$$
e^{-j 2 \pi \cdot 1 \cdot 0 / 4} = e^{0} = 1
$$

- For $n=1$:

$$
e^{-j 2 \pi \cdot 1 \cdot 1 / 4} = e^{-j \pi/2} = \cos(-\pi/2) + j \sin(-\pi/2) = 0 - j1 = -j
$$

- For $n=2$:

$$
e^{-j 2 \pi \cdot 1 \cdot 2 / 4} = e^{-j \pi} = -1
$$

- For $n=3$:

$$
e^{-j 2 \pi \cdot 1 \cdot 3 / 4} = e^{-j 3\pi/2} = \cos(-3\pi/2) + j \sin(-3\pi/2) = 0 + j1 = j
$$

Now, multiply each by corresponding $x(n)$:

$$
X(1) = 1 \cdot 1 + 3 \cdot (-j) + 3 \cdot (-1) + 4 \cdot j
$$

Calculating:

$$
X(1) = 1 - 3j - 3 + 4j = (1 - 3) + (-3j + 4j) = -2 + j
$$

Computing $X(2)$

$$
X(2) = \sum_{n=0}^{3} x(n) \cdot e^{-j 2 \pi \cdot 2 \cdot n / 4}
$$

Exponentials:

- $n=0$:

$$
e^{0} = 1
$$

- $n=1$:

$$
e^{-j \pi} = -1
$$

- $n=2$:

$$
e^{-j 2\pi} = 1
$$

- $n=3$:

$$
e^{-j 3\pi} = -1
$$

Multiplying:

$$
X(2) = 1 \cdot 1 + 3 \cdot (-1) + 3 \cdot 1 + 4 \cdot (-1)
$$

Calculating:

$$
X(2) = 1 - 3 + 3 - 4 = (1 - 3 + 3 - 4) = -3
$$

Computing $X(3)$

$$
X(3) = \sum_{n=0}^{3} x(n) \cdot e^{-j 2 \pi \cdot 3 \cdot n / 4}
$$

Exponentials:

- $n=0$:

$$
e^{0} = 1
$$

- $n=1$:

$$
e^{-j 3\pi/2} = 0 + j1 = j
$$

- $n=2$:

$$
e^{-j 3\pi} = e^{j \pi} = -1
$$

- $n=3$:

$$
e^{-j 9\pi/2} = e^{-j (4.5\pi)} = e^{-j 4\pi} \cdot e^{-j \pi/2} = 1 \cdot (-j) = -j
$$

Note: Since $e^{-j 2\pi m} = 1$ for integer m, and $e^{-j 4\pi} = 1$.

Calculations:

$$
X(3) = 1 \cdot 1 + 3 \cdot j + 3 \cdot (-1) + 4 \cdot (-j)
$$

Simplify:

$$
X(3) = 1 + 3j - 3 - 4j = (1 - 3) + (3j - 4j) = -2 - j
$$

Summary of Calculated DFT Components

k	$X(k)$	Complex Form
0	11	11 + 0j
1	-2 + j	-2 + j
2	-3	-3 + 0j
3	-2 - j	-2 - j

Interpretation of Results

The computed DFT results provide insight into the frequency content of the sequence:

- $X(0)$: The sum of all sequence elements, representing the average or DC component.
- $X(1)$ and $X(3)$: Complex conjugates, indicating symmetric frequency components.
- $X(2)$: Represents the frequency component at half the sampling rate.

The magnitude and phase of these components reflect the signal's spectral characteristics. For example, the magnitude of $X(1)$ is:

$$
|X(1)| = \sqrt{(-2)^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5} \approx 2.236
$$

Similarly, phase angles can be calculated to understand signal shifts.

Applications of 4-point DFT

The 4-point DFT is a fundamental building block in digital signal processing, with applications including:

- Spectral analysis: Identifying dominant frequencies in signals.
- Filtering: Designing filters in the frequency domain.
- Signal compression: Reducing data by retaining significant frequency components.
- Image processing: Extending Fourier techniques to two dimensions.

Practical Implementation Tips

When implementing DFT calculations:

- Use software tools like MATLAB, Python (NumPy), or specialized DSP hardware for efficiency.
- Be mindful of numerical precision, especially with complex exponential calculations.
- For longer sequences, Fast Fourier Transform (FFT) algorithms significantly reduce computational complexity.

Conclusion