5. Let R(t)=(cost,sint,t). A. Find The Unit Tangent Vector T. B. Find The Unit Normal Vector N. Hint.

5. Let R(t) = (cos t, sin t, t). A. Find The Unit Tangent Vector T. B. Find The Unit Normal Vector N. Hint.

In vector calculus and differential geometry, understanding the behavior of curves in space is fundamental. The position vector R(t) = (cos t, sin t, t) describes a space curve that spirals upward around the z-axis, combining circular motion in the xy-plane with linear motion along the z-axis. To analyze this curve's geometric properties—such as its direction, curvature, and how it bends—it is essential to compute the unit tangent and normal vectors. These vectors provide insight into the curve's orientation at any point and are critical for understanding the curve's shape and movement.

This article will thoroughly walk through the process of finding the unit tangent vector T and the unit normal vector N for the given vector function R(t). We will explore the definitions, step-by-step calculations, and the geometric implications of these vectors, including useful hints and tips to facilitate understanding.

Understanding the Curve R(t) = (cos t, sin t, t)

Before diving into calculations, it's helpful to visualize the curve:

- In the xy-plane: The projection of R(t) onto the xy-plane traces a circle of radius 1 as t varies, since x = cos t and y = sin t.

- Along the z-axis: The z-coordinate increases linearly with t, creating a spiral or helix that wraps around the z-axis.

This combination results in a classic helix that spirals upward, a common subject in differential geometry due to its simplicity and symmetry.

Part A: Finding the Unit Tangent Vector T

Definition of the Unit Tangent Vector

The unit tangent vector T(t) at a point R(t) on a curve is defined as:

$$
T(t) = \frac{R'(t)}{\| R'(t) \|},
$$

where:

- $R'(t)$ is the derivative of the position vector with respect to t.

- $\| R'(t) \|$ is the magnitude (or norm) of $R'(t)$.

The vector $R'(t)$ points in the direction of the curve’s motion at t, and dividing it by its magnitude normalizes it to a unit vector.

Calculating R'(t)

Given:

$$
R(t) = (\cos t, \sin t, t),
$$

we differentiate component-wise:

$$
R'(t) = \left(-\sin t, \cos t, 1 \right).
$$

This derivative represents the velocity vector at parameter t.

Computing the Magnitude $\| R'(t) \|$

The magnitude is:

$$
\| R'(t) \| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1}.
$$

Since $\sin^2 t + \cos^2 t = 1$, we get:

$$
\| R'(t) \| = \sqrt{1 + 1} = \sqrt{2}.
$$

Forming the Unit Tangent Vector T(t)

Dividing each component of $R'(t)$ by $\sqrt{2}$:

$$
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1).
$$

This vector points in the direction of the curve's motion at t and has magnitude 1.

Summary for Part A

$$
\boxed{
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1).
}
$$

This unit tangent vector provides the direction of the curve at any point t, normalized to length 1, making it invaluable for further analysis like curvature or normal vectors.

Part B: Finding the Unit Normal Vector N

Understanding the Unit Normal Vector

The unit normal vector N(t) is perpendicular to T(t) and points toward the center of curvature at a given point on the curve. It indicates the direction in which the curve is turning and is essential for understanding the curvature and the "bend" of the curve.

Mathematically, N(t) can be obtained by differentiating T(t) with respect to t and normalizing the result:

$$
N(t) = \frac{T'(t)}{\| T'(t) \|}.
$$

This process involves computing the derivative of T(t), then normalizing to ensure it is a unit vector.

Calculating T'(t)

Recall:

$$
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1).
$$

Differentiating component-wise:

$$
T'(t) = \frac{1}{\sqrt{2}} (-\cos t, -\sin t, 0).
$$

Note that the derivative of the constant component (the third component, 1) is zero, and the derivatives of $-\sin t$ and $\cos t$ are $-\cos t$ and $-\sin t$, respectively.

Calculating $\| T'(t) \|$

The magnitude of $T'(t)$ is:

$$
\| T'(t) \| = \frac{1}{\sqrt{2}} \sqrt{ (-\cos t)^2 + (-\sin t)^2 + 0^2 } = \frac{1}{\sqrt{2}} \sqrt{ \cos^2 t + \sin^2 t }.
$$

Since $\cos^2 t + \sin^2 t = 1$:

$$
\| T'(t) \| = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}}.
$$

Forming the Unit Normal Vector N(t)

Dividing $T'(t)$ by its magnitude:

$$
N(t) = \frac{T'(t)}{\| T'(t) \|} = \frac{ \frac{1}{\sqrt{2}} (-\cos t, -\sin t, 0) }{ \frac{1}{\sqrt{2}} }.
$$

The $\frac{1}{\sqrt{2}}$ factors cancel:

$$
N(t) = (-\cos t, -\sin t, 0).
$$

This vector is already of unit length, since:

$$
\| N(t) \| = \sqrt{ \cos^2 t + \sin^2 t + 0 } = 1.
$$

Summary for Part B

$$
\boxed{
N(t) = (-\cos t, -\sin t, 0).
}
$$

This unit normal vector points toward the center of the circle in the xy-plane and indicates the direction in which the curve is curving at each point.

Additional Insights and Geometric Interpretation

- **Curve Geometry:** The space curve R(t) = (cos t, sin t, t) is a helix, wrapping around the z-axis with constant radius 1 and ascending linearly along z.
- **Tangent vector T(t):** Points in the direction of motion along the helix, combining horizontal and vertical components.
- **Normal vector N(t):** Points toward the center of the helix's curvature in the xy-plane, indicating the direction in which the curve is "turning."
- **Curvature and Torsion:** The unit tangent and normal vectors are essential for calculating the curvature (κ) and torsion (τ), which describe how sharply the curve bends and twists.

Summary of Results

Vector	Expression	Description
Unit Tangent Vector T(t)	$\frac{1}{\sqrt{2}} (-\sin t, \cos t, 1)$	Direction of the curve at t, normalized to length 1.

| Unit Normal Vector N(t) | \((- \cos t, - \sin t, 0)\) | Direction of the curve's bending in the xy-plane, normalized. |

Conclusion

The process of finding the unit tangent and normal vectors for a space curve like R(t) = (cos t, sin t, t) is fundamental in understanding its geometric properties. The unit tangent vector T(t) indicates the direction of motion at each point, while the unit normal vector N(t) reveals how the curve bends. These vectors are foundational for further analysis, including curvature, torsion, and the Frenet-Serret frame, which collectively describe the shape and behavior of space curves.

Understanding these vectors not only enhances comprehension of the specific helix but also provides a template for analyzing more complex curves in three

Frequently Asked Questions

What is the given vector function R(t) in the problem?

The vector function is R(t) = (cos t, sin t, t).

How do you find the unit tangent vector T for R(t)?

First, find the derivative R'(t), then normalize it by dividing by its magnitude: T = R'(t)/|R'(t)|.

What is the derivative R'(t) of R(t) = (cos t, sin t, t)?

R'(t) = (-sin t, cos t, 1).

How do you compute the magnitude |R'(t)|?

Calculate |R'(t)| = sqrt[(-sin t)^2 + (cos t)^2 + 1^2] = sqrt(sin^2 t + cos^2 t + 1) = sqrt(1 + 1) = sqrt(2).

What is the unit tangent vector T for R(t)?

T = R'(t)/|R'(t)| = (-sin t, cos t, 1) / sqrt(2).

How do you find the unit normal vector N to the curve?

Compute the derivative T'(t), then normalize it: N = T'(t)/|T'(t)|.

What is T'(t) for the given R(t)?

Since T(t) = (-sin t / sqrt{2}, cos t / sqrt{2}, 1 / sqrt{2}), its derivative is T'(t) = (-cos t / sqrt{2}, -sin t / sqrt{2}, 0).

How do you find the magnitude |T'(t)|?

Calculate |T'(t)| = sqrt[(-cos t / sqrt{2})^2 + (-sin t / sqrt{2})^2 + 0^2] = sqrt[(cos^2 t + sin^2 t)/2] = sqrt(1/2) = 1/√2.

What is the unit normal vector N at time t?

N = T'(t)/|T'(t)| = (-cos t / sqrt{2}, -sin t / sqrt{2}, 0) divided by 1/√2, which simplifies to (-cos t, -sin t, 0).

What is the significance of the vectors T and N in the context of the curve R(t)?

T is the unit tangent vector indicating the direction of the curve at t, while N is the principal normal vector pointing towards the center of curvature, both describing the curve's geometric properties.

Additional Resources

Vector Calculus Analysis: Exploring the Curve R(t) = (cos t, sin t, t)

Introduction: Unveiling the Geometric Essence of R(t)

In the realm of vector calculus and differential geometry, parametric curves serve as the foundational building blocks to understanding the shape and structure of space curves. Among these, the curve defined by the vector function

$$R(t) = (\cos t, \sin t, t)$$

stands out as a classic example that elegantly combines circular motion with a linear progression along the z-axis. Often described as a helix, this curve encapsulates the harmony between periodicity and linearity, making it an ideal candidate for exploring fundamental concepts such as tangent vectors, normal vectors, curvature, and torsion.

In this comprehensive analysis, we'll delve into the details of this curve, focusing particularly on:

- Deriving the unit tangent vector $T(t)$
- Determining the unit normal vector $N(t)$

This exploration not only illuminates the geometric properties of the curve but also provides insight into the underlying calculus techniques essential for advanced studies in fields ranging from physics to computer graphics.

Part A: Finding the Unit Tangent Vector $T(t)$

Understanding the Tangent Vector Concept

The tangent vector at a point on a curve indicates the direction in which the curve proceeds. For a parametric curve $R(t)$, the tangent vector is obtained by differentiating the position vector with respect to the parameter t:

$$R'(t) = \frac{d}{dt} R(t)$$

However, to analyze the geometric properties independent of the curve's speed, we normalize this derivative to produce a unit tangent vector:

$$T(t) = \frac{R'(t)}{|R'(t)|}$$

This vector points in the direction of the curve's motion at each point and has unit length, making it a fundamental tool in differential geometry.

Step-by-Step Calculation of $T(t)$

Given:

$$R(t) = (\cos t, \sin t, t)$$

Let's differentiate each component with respect to t:

$$
\begin{cases}
\frac{d}{dt} \cos t = -\sin t \\
\frac{d}{dt} \sin t = \cos t \\
\frac{d}{dt} t = 1
\end{cases}
$$

Thus,

$$ R'(t) = (-\sin t, \cos t, 1) $$

Next, compute the magnitude $|R'(t)|$:

$$
|R'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1}
$$

Recall the Pythagorean identity:

$$ \sin^2 t + \cos^2 t = 1 $$

Therefore,

$$
|R'(t)| = \sqrt{1 + 1} = \sqrt{2}
$$

The unit tangent vector $T(t)$ is then:

$$
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1)
$$

Final expression:

$$
\boxed{
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1)
}
$$

Interpretation:

- The x and y components oscillate sinusoidally, reflecting the circular motion.
- The z-component remains constant at $1/\sqrt{2}$, indicating a uniform upward tilt, characteristic of a helix.

Part B: Finding the Unit Normal Vector $N(t)$

The Significance of the Normal Vector

While the unit tangent vector $T(t)$ points along the direction of the curve, the unit normal vector $N(t)$ points towards the center of the curve's instantaneous curvature, indicating how the curve bends. It is perpendicular to $T(t)$ and lies in the osculating plane, providing a local "direction of curvature."

Mathematically, the normal vector is derived from the derivative of $T(t)$:

$$ N(t) = \frac{T'(t)}{|T'(t)|} $$

This process involves differentiating $T(t)$, normalizing the result, and ensures that $N(t)$ is a unit vector orthogonal to $T(t)$.

Calculating $N(t)$

Step 1: Compute $T'(t)$

Recall:

$$
T(t) = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1)
$$

Differentiate component-wise:

$$
\begin{cases}
\frac{d}{dt} \left(-\frac{\sin t}{\sqrt{2}} \right) = -\frac{\cos t}{\sqrt{2}} \\
\frac{d}{dt} \left(\frac{\cos t}{\sqrt{2}} \right) = -\frac{\sin t}{\sqrt{2}} \\
\frac{d}{dt} \left(\frac{1}{\sqrt{2}} \right) = 0
\end{cases}
$$

Therefore,

$$
T'(t) = \left(-\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right)
$$

Step 2: Find the magnitude $|T'(t)|$

$$
|T'(t)| = \sqrt{\left(-\frac{\cos t}{\sqrt{2}}\right)^2 + \left(-\frac{\sin t}{\sqrt{2}}\right)^2 + 0^2}
$$

Simplify:

$$
|T'(t)| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{\cos^2 t + \sin^2 t}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}
$$

Step 3: Normalize \(T'(t) \) to obtain \(N(t) \)

\[
N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\left(-\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right)}{\frac{1}{\sqrt{2}}}
\]

Dividing numerator by denominator:

\[
N(t) = (-\cos t, -\sin t, 0)
\]

Final expression:

\[
\boxed{
N(t) = (-\cos t, -\sin t, 0)
}
\]

Interpretation:

- \(N(t) \) points radially inward in the \(xy \)-plane, directly opposite to the position vector \((\cos t, \sin t) \).
- It is perpendicular to \(T(t) \), satisfying the orthogonality condition \(T(t) \cdot N(t) = 0 \).

Concluding Insights and Geometric Intuition

The parametric curve \(R(t) = (\cos t, \sin t, t) \) embodies a right-handed helix—a spiral that ascends along the \(z \)-axis while revolving in the \(xy \)-plane. Its geometric features are vividly captured by its tangent and normal vectors:

- The unit tangent vector \(T(t) \) combines the circular motion's direction with a constant upward component, emphasizing the smooth, continuous progression along the helix. Its normalized form ensures that the vector's magnitude remains unity, facilitating consistent geometric analysis.

- The unit normal vector \(N(t) \) points towards the center of curvature in the \(xy \)-plane, reflecting how the curve "bends" around the axis. Its direction, directly opposite to the radial position vector, aligns with the intuitive understanding that the curve curves inward as it wraps around the axis.

This detailed analysis not only underscores the elegance of the mathematical structures underlying space curves but also exemplifies the power of vector calculus techniques in revealing the intrinsic geometry of complex shapes.

Additional Considerations: Curvature and Torsion

While the focus here is on the tangent and normal vectors, further exploration involves calculating:

- Curvature $\kappa(t)$, which measures how sharply the curve bends.
- Torsion $\tau(t)$, which quantifies the rate of twisting out of the osculating plane.

These quantities deepen the understanding of the helix's geometric nature and are essential in applications such as the analysis of DNA structures, robotic path planning, and computer-aided design.

Final Remarks

The parametric curve $R(t) = (\cos t, \sin t, t)$ offers a rich tapestry for applying fundamental vector calculus tools. By meticulously deriving the unit tangent and normal vectors, we've uncovered the geometric heartbeat of the hel

[5 Let R T Cost Sint T A Find The Unit Tangent Vector T B Find The Unit Normal Vector N Hint]

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