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Understanding the behavior and properties of random samples is fundamental in probability theory and statistical inference. In this section, we delve into the scenario where the random variables $X, X_2, \dots, X_n$ are drawn independently from a discrete distribution defined over the positive integers, with a specific probability mass function (pmf). We analyze the implications of such sampling, focusing on expectations, variance, and related statistical measures.

Overview of the Distribution

Definition of the Distribution

The problem specifies a discrete probability distribution for the random variable X:

- $P_X(x) = P(X = x) = p_x$, where $x = 1, 2, 3, \dots$.
- $P_X(x) = 0$ otherwise.

This indicates that the support of X is the set of positive integers $\{1, 2, 3, \dots \}$.

Conditions on the Probabilities

Since this is a valid probability distribution, the following conditions hold:

1. $p_x \geq 0$ for all $x \geq 1$
2. $\sum_{x=1}^\infty p_x = 1$

The specific form of p_x determines the nature of the distribution, such as whether it is geometric, negative binomial, or another discrete distribution.

Expected Value $E[X]$

Definition of Expectation

The expectation of X, denoted $E[X]$, measures the long-run average value of the random variable over many independent samples:

$$
E[X] = \sum_{x=1}^\infty x \, p_x
$$

This sum converges when the probabilities decay sufficiently fast, ensuring a finite mean.

Importance of $E[X]$

The expected value provides insights into the central tendency of the distribution:

- It helps in predicting the average outcome of the random process.
- It serves as a basis for variance, standard deviation, and other moments.
- In inferential statistics, it forms the basis for estimators and hypothesis tests.

Sampling from the Distribution

Independent and Identically Distributed (i.i.d.) Samples

Assuming $X_1, X_2, \dots, X_n$ are i.i.d. random variables with the same distribution as X:

- Each X_i has the same pmf p_X.
- Samples are independent, meaning the occurrence of one does not influence others.

Sample Mean and Its Properties

The sample mean:

$$
\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i
$$

has properties linked to the population mean $E[X]$:

- By the Law of Large Numbers (LLN), as $n \to \infty$, $\bar{X}_n \to E[X]$ almost surely.
- The expectation of the sample mean is $E[\bar{X}_n] = E[X]$.
- The variance of the sample mean:

  $$
  Var(\bar{X}_n) = \frac{Var(X)}{n}
  $$

  which decreases as n increases, implying more precise estimates of $E[X]$ with larger samples.

Calculating the Expectation for Specific Distributions

Geometric Distribution

One common discrete distribution over positive integers is the geometric distribution, where:

$$
p_x = (1 - p)^{x-1} p, \quad 0 < p < 1
$$

The expectation:

$$
E[X] = \frac{1}{p}
$$

and the variance:

$$
Var[X] = \frac{1 - p}{p^2}
$$

This distribution models the number of trials until the first success.

Negative Binomial Distribution

The negative binomial distribution generalizes the geometric distribution, representing the number of trials needed to achieve a fixed number of successes.

- Its pmf:

$$
p_x = \binom{x-1}{r-1} p^r (1 - p)^{x - r}
$$

where r is the number of successes, and p the success probability per trial.

- Expectation:

$$
E[X] = \frac{r}{p}
$$

- Variance:

$$
Var[X] = \frac{r(1 - p)}{p^2}
$$

Implications and Applications

Estimating $E[X]$ from Sample Data

In practical scenarios, understanding $E[X]$ helps in:

1. Predictive modeling: estimating average outcomes in processes modeled by the distribution.

2. Quality control: determining average number of defects or failures.
3. Risk assessment: understanding expected losses or failures over time.

Confidence Intervals for $E[X]$

Given a sample $X_1, X_2, \dots, X_n$, the sample mean $\bar{X}_n$ approximates $E[X]$. To quantify uncertainty, confidence intervals can be constructed using:

- Central Limit Theorem (CLT) for large n, assuming finite variance.
- Student's t-distribution for small or moderate n, especially if variance is estimated from the sample.

Conclusion

Sampling from a discrete distribution with support over positive integers provides foundational insights into the behavior of such random variables. The expectation $E[X]$ is a critical measure, influencing predictions, statistical inference, and decision-making processes. Whether dealing with geometric, negative binomial, or other distributions, understanding their properties and how to estimate $E[X]$ from data is essential in numerous scientific, engineering, and economic applications.

By thoroughly analyzing the distribution's pmf, calculating expectations, and understanding sampling implications, statisticians and researchers can make informed decisions based on the data collected from such discrete random processes.

Frequently Asked Questions

What is the distribution of the random variables $X_1, X_2, \dots, X_n$ in the given setup?

They are independent and identically distributed (i.i.d.) random variables drawn from a distribution with probability mass function $P_n(x) = (1 - p) p^{x-1}$ for $x = 1, 2, 3, \dots$, and 0 otherwise, which is a geometric distribution with parameter p.

How do we interpret the expected value E[X] for the geometric distribution described?

The expected value $E[X] = 1 / p$, representing the average number of trials needed to get the first success in a geometric process with success probability p.

What is the significance of the parameter p in this geometric distribution?

The parameter p represents the probability of success on each independent trial, influencing the shape and mean of the distribution; as p increases, the expected value decreases.

How can the properties of this geometric distribution be used in statistical inference?

Properties like the mean and variance allow estimation of the success probability p and modeling of processes like waiting times or failure rates in reliability and survival analysis.

What are common applications of the geometric distribution in real-world scenarios?

It is used in modeling the number of trials until the first success in quality control, reliability testing, biological experiments, and in modeling waiting times in queueing systems.

Additional Resources

Understanding the Distribution of a Discrete Random Variable and Its Expected Value: An In-Depth Analysis

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Introduction

In the realm of probability theory and statistics, understanding the behavior of random variables is fundamental. These variables serve as mathematical models for random phenomena, allowing us to quantify uncertainty, make predictions, and infer properties about underlying processes. Among various types of random variables, discrete random variables hold a prominent place due to their applicability in numerous real-world scenarios—from modeling the number of emails received in a day to the count of defective items in a batch.

This article explores a particular class of discrete random variables characterized by a specific probability mass function (pmf), namely the geometric-like distribution specified as:

$$P_X(x) = \begin{cases}
(1 - p) p^{x - 1}, & x = 1, 2, 3, \ldots \\
0, & \text{otherwise}
\end{cases}$$

where $0 < p < 1$. We will analyze this distribution comprehensively, focusing on its properties, expectation, and implications for statistical inference. The discussion is structured to provide clarity, depth, and insight into the distribution's behavior and its role in modeling random phenomena.

Defining the Distribution: The Geometric Distribution

Formulation and Intuition

The pmf

$$P_X(x) = (1 - p) p^{x - 1} \quad \text{for } x = 1, 2, 3, \ldots$$

describes a geometric distribution with parameter p. Intuitively, this distribution models the number of trials required to achieve the first success in a sequence of independent Bernoulli trials, each with success probability p.

- Key characteristics:
- The variable X takes on positive integer values, starting from 1.
- The probability decreases geometrically as x increases, governed by the factor p^{x-1}.
- The distribution is memoryless, meaning the probability of success in future trials does not depend on past

failures.

Historical Context and Applications

The geometric distribution has roots in the early development of probability theory, notably studied by the mathematician Jacob Bernoulli. Today, it finds applications in various fields:

- Quality control: Counting the number of items inspected until a defective is found.
- Reliability engineering: Time (or number of trials) until a system failure.
- Biology: Number of generations until a mutation occurs.
- Computer science: Modeling the number of attempts until a correct guess.

Mathematical Properties of the Distribution

Normalization and Validity of the pmf

A fundamental property of probability distributions is that the total probability sums to 1:

$$
\sum_{x=1}^{\infty} P_X(x) = \sum_{x=1}^{\infty} (1 - p) p^{x-1} = (1 - p) \sum_{x=0}^{\infty} p^{x} = (1 - p) \frac{1}{1 - p} = 1
$$

since $0 < p < 1$. This confirms the pmf is valid, satisfying the axioms of probability.

Expected Value (Mean)

The expected value $E[X]$ of a geometric distribution is a cornerstone parameter, capturing the average number of trials until the first success:

$$
E[X] = \sum_{x=1}^{\infty} x P_X(x)
$$

Calculating this involves summing an infinite series:

$$
E[X] = (1 - p) \sum_{x=1}^{\infty} x p^{x-1}
$$

Recognizing the sum as a well-known series, we can derive:

$$
\sum_{x=1}^{\infty} x r^{x} = \frac{r}{(1 - r)^2} \quad \text{for } |r| < 1
$$

Substituting $r = p$:

\[

$$
E[X] = (1 - p) \times \frac{1}{(1 - p)^2} = \frac{1}{1 - p}
$$

Thus,

$$
\boxed{
E[X] = \frac{1}{p}
}
$$

This simple yet powerful result indicates that the expected number of trials until the first success is inversely proportional to the success probability p.

Variance

The variance measures the dispersion of the distribution:

$$
Var(X) = E[X^2] - (E[X])^2
$$

Calculating $E[X^2]$:

$$
E[X^2] = \sum_{x=1}^{\infty} x^2 P_X(x) = (1 - p) \sum_{x=1}^{\infty} x^2 p^{x-1}
$$

Using the known sum:

$$
\sum_{x=1}^{\infty} x^2 r^{x} = \frac{r (1 + r)}{(1 - r)^3}
$$

Replacing $r = p$:

$$
E[X^2] = (1 - p) \times \frac{p (1 + p)}{(1 - p)^3} = \frac{p (1 + p)}{(1 - p)^2}
$$

Therefore,

$$
Var(X) = \frac{p (1 + p)}{(1 - p)^2} - \left(\frac{1}{p} \right)^2 = \frac{p (1 + p)}{(1 - p)^2} - \frac{1}{p^2}
$$

Simplifying yields:

$$
Var(X) = \frac{1 - p}{p^2}
$$

Sampling and Estimation: From Theory to Practice

Sampling from the Distribution

Suppose $X_1, X_2, \ldots, X_n$ form a random sample drawn independently from the geometric distribution with parameter p. This sampling process is crucial for statistical inference, allowing us to estimate the parameter p and understand the distribution empirically.

Sampling techniques often include:

- Inverse transform sampling: Using the inverse of the cumulative distribution function (CDF).
- Rejection sampling: Less common for geometric variables but applicable in complex scenarios.
- Simulation algorithms: Implemented in statistical software to generate random variates.

Maximum Likelihood Estimation (MLE) of p

Given observed data, estimating p accurately is central. The likelihood function for the sample is:

$$
L(p) = \prod_{i=1}^{n} P_X(x_i) = \prod_{i=1}^{n} (1 - p) p^{x_i - 1}
$$

The log-likelihood:

$$
\ell(p) = n \ln(1 - p) + \left(\sum_{i=1}^n (x_i - 1) \right) \ln p
$$

Maximizing $\ell(p)$ with respect to p involves setting the derivative to zero:

$$
\frac{d\ell(p)}{dp} = - \frac{n}{1 - p} + \frac{\sum_{i=1}^n (x_i - 1)}{p} = 0
$$

Solving for p:

$$
p = \frac{n}{n + \sum_{i=1}^n (x_i - 1)} = \frac{n}{\sum_{i=1}^n x_i}
$$

Thus, the MLE for p:

$$
\boxed{
\hat{p} = \frac{n}{\sum_{i=1}^n x_i}
}
$$

This estimator is intuitive: the average of the observed x_i's is $1 / p$, aligning with the theoretical expected value.

Implications and Limitations

Model Assumptions and Real-World Considerations

While the geometric distribution offers elegant and straightforward properties, applying it to real data requires caution:

- Memoryless property: Not all phenomena exhibit this; for example, in process durations with aging effects.
- Parameter constraints: p must lie strictly between 0 and 1.
- Overdispersion: If data variance exceeds the mean, alternative models like negative binomial may be more appropriate.

Extensions and Variations

The basic geometric distribution can be extended or modified:

- Shifted geometric: Starts from 0 instead of 1.
- Negative binomial: Counts the number of trials until the r-th success.
- Zero-truncated models: Adjusted for scenarios where zero counts are impossible.

Conclusion

The geometric distribution characterized by the pmf $P_X(x) = (1 - p) p^{x - 1}$ embodies a fundamental concept in probability: modeling the waiting time until the first success. Its properties—such

5 Let X X2 Xn Be A Random Sample From 1 0 0 Px X X 1 2 3 0 Otherwise Where E X

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