8. The Area Of The Parallelogram Whose Adjacent Sides Formed By The Vectors Usi+i-k And V= 2i-j+3k Is

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Understanding the concept of the area of a parallelogram in vector calculus is fundamental in various fields such as physics, engineering, and mathematics. When the adjacent sides of a parallelogram are represented by vectors, the area can be computed using the magnitude of their cross product. This article explores the detailed process of calculating the area of a parallelogram formed by the vectors \(\mathbf{U} = s\mathbf{i} + \mathbf{i} - \mathbf{k} \) and \(\mathbf{V} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \).

Understanding Vectors and Their Representation

What Are Vectors?

Vectors are quantities that have both magnitude and direction. They are represented mathematically in three-dimensional space using components along the x, y, and z axes, typically denoted as \(\mathbf{i} \), \(\mathbf{j} \), and \(\mathbf{k} \), respectively.

Component Form of Vectors

- Vector \(\mathbf{U} \): Defined as \(\mathbf{U} = s\mathbf{i} + \mathbf{i} - \mathbf{k} \).
- Vector \(\mathbf{V} \): Given as \(\mathbf{V} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \).

The component form allows for easy calculation of cross products and dot products necessary for the area computation.

Expressing Vectors in Standard Component Form

Vector \(\mathbf{U} \)

- The \(\mathbf{i} \) component: \(s\mathbf{i} + \mathbf{i} = (s + 1) \mathbf{i} \).
- The \(\mathbf{j} \) component: Since there is no \(\mathbf{j} \) term explicitly, it is 0.
- The \(\mathbf{k} \) component: \(-\mathbf{k} = -1 \).

Therefore, in component form:

$$
\mathbf{U} = (s + 1) \mathbf{i} + 0 \mathbf{j} - 1 \mathbf{k} \Rightarrow
\mathbf{U} = (s + 1,\, 0,\, -1)
$$

Vector $\mathbf{V}$

- The $\mathbf{i}$ component: 2
- The $\mathbf{j}$ component: -1
- The $\mathbf{k}$ component: 3

Expressed as:

$$
\mathbf{V} = (2,\, -1,\, 3)
$$

Calculating the Cross Product of Two Vectors

The cross product of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ is given by:

$$
\mathbf{A} \times \mathbf{B} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
$$

which expands to:

$$
\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \mathbf{i} - (A_x B_z - A_z B_x) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k}
$$

Applying this to our vectors, we get:

$$
\mathbf{U} \times \mathbf{V} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
s + 1 & 0 & -1 \\
2 & -1 & 3
\end{vmatrix}
$$

Computing the Cross Product Step-by-Step

Step 1: Calculate the $\mathbf{i}$-component

$$
(0)(3) - (-1)(-1) = 0 - 1 = -1
$$

Step 2: Calculate the $\mathbf{j}$-component

$$
-[(s+1)(3) - (-1)(2)] = -[(3s + 3) - (-2)] = -[3s + 3 + 2] = -[3s + 5]
$$

Step 3: Calculate the $\mathbf{k}$-component

$$
(s+1)(-1) - (0)(2) = - (s+1) - 0 = -s - 1
$$

Putting it all together, the cross product vector is:

$$
\mathbf{U} \times \mathbf{V} = (-1) \mathbf{i} + (3s + 5) \mathbf{j} - (s + 1) \mathbf{k}
$$

or in component form:

$$
\mathbf{U} \times \mathbf{V} = (-1,\, 3s + 5,\, -s - 1)
$$

Finding the Magnitude of the Cross Product

The magnitude of the cross product vector gives the area of the parallelogram formed by the two vectors:

$$
\text{Area} = |\mathbf{U} \times \mathbf{V}| = \sqrt{(-1)^2 + (3s + 5)^2 + (-s - 1)^2}
$$

Simplify step-by-step:

$$
|\mathbf{U} \times \mathbf{V}| = \sqrt{1 + (3s + 5)^2 + (s + 1)^2}
$$

Note that $-s - 1$ squared is the same as $(s + 1)^2$, since squaring removes the negative sign.

Expanding and Simplifying the Expression

Step 1: Expand the squared terms

$$
(3s + 5)^2 = 9s^2 + 30s + 25
$$

$$
(s + 1)^2 = s^2 + 2s + 1
$$

Step 2: Sum all parts inside the square root

$$
1 + 9s^2 + 30s + 25 + s^2 + 2s + 1
$$

Combine like terms:

- For s^2: $9s^2 + s^2 = 10s^2$
- For s: $30s + 2s = 32s$
- Constant terms: $1 + 25 + 1 = 27$

Thus, the magnitude simplifies to:

$$
|\mathbf{U} \times \mathbf{V}| = \sqrt{10s^2 + 32s + 27}
$$

Final Expression for the Area of the Parallelogram

The area of the parallelogram formed by the vectors $\mathbf{U}$ and $\mathbf{V}$ is:

$$
\boxed{
\text{Area} = \sqrt{10s^2 + 32s + 27}
}
$$

This formula indicates that the area depends on the parameter s, which can be varied to find specific areas for different values of s.

Special Cases and Additional Insights

When s is a specific value

- For example, if $s = 0$:
$$
\text{Area} = \sqrt{10(0)^2 + 32(0) + 27} = \sqrt{27} = 3\sqrt{3}
$$
- If $s = -\frac{16}{10} = -\frac{8}{5}$ (to make the quadratic zero), the area reduces to zero, indicating that the vectors become linearly dependent, and no parallelogram is formed.

Implications of the formula

- The formula provides a direct way to compute the area for any value of s, offering flexibility in applications where s might vary.
- The dependency of the area on s can be analyzed to understand how changing the parameter affects the size of the parallelogram.

Conclusion

Calculating the area of a parallelogram in three-dimensional space using vectors involves understanding the cross product and its magnitude. By expressing the vectors in component form, performing the cross product, and simplifying, we derive a formula that depends on the parameter s. The key steps include:

- Expressing the given vectors in component form.
- Computing the cross product using the determinant method

Frequently Asked Questions

How do you find the area of a parallelogram formed by two vectors U and V in three-dimensional space?

The area of the parallelogram is given by the magnitude of the cross product of the two vectors, i.e., $|U \times V|$.

What are the vectors U and V given in the problem?

$U = i + k - i$, which simplifies to $U = 0i + 1j + 1k$, and $V = 2i - j + 3k$.

How do you compute the cross product $U \times V$ for the given vectors?

Using the determinant method, set up a 3x3 matrix with unit vectors i, j, k in the first row, components of U in the second, and components of V in the third, then compute its determinant.

What is the cross product $U \times V$ for the vectors $U = 0i + 1j + 1k$ and $V = 2i - j + 3k$?

$U \times V = ((1)(3) - (1)(-1))i - ((0)(3) - (1)(2))j + ((0)(-1) - (1)(2))k = (3 + 1)i - (0 - 2)j + (0 - 2)k = 4i + 2j - 2k$.

How do you calculate the magnitude of the cross product to find the area?

The magnitude is calculated as $\sqrt{4^2 + 2^2 + (-2)^2} = \sqrt{16 + 4 + 4} = \sqrt{24} = 2\sqrt{6}$.

What is the final area of the parallelogram?

The area is $2\sqrt{6}$ square units.

Why is the magnitude of the cross product important in this context?

It directly gives the area of the parallelogram formed by the two vectors in three-dimensional space.

Additional Resources

8. The Area Of The Parallelogram Whose Adjacent Sides Formed By The Vectors Usi+i-k And V= 2i-j+3k Is

In the realm of vector calculus, understanding the geometric interpretation of vectors and their operations is crucial. One fascinating application is determining the area of a parallelogram formed by two vectors in three-dimensional space. This problem not only exemplifies fundamental concepts in vector algebra but also showcases how algebraic operations translate into geometric properties. Today, we delve into the specifics of calculating the area of a parallelogram whose adjacent sides are defined by the vectors $\mathbf{U} = s\mathbf{i} + \mathbf{i} - \mathbf{k}$ and $\mathbf{V} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Our goal is to interpret these vectors, understand the mathematical tools involved, and derive the exact area of the parallelogram they form.

Understanding the Vectors and Their Components

Before embarking on the calculation, it’s vital to comprehend what these vectors represent and how they are expressed in coordinate form.

Deciphering Vector U

The vector \(\mathbf{U} \) is given as:

\[
\mathbf{U} = s\mathbf{i} + \mathbf{i} - \mathbf{k}
\]

At first glance, this expression combines scalar and vector components, but it can be simplified into a standard component form. Recognizing that \(\mathbf{i} \) corresponds to the unit vector along the x-axis, \(\mathbf{j} \) along the y-axis, and \(\mathbf{k} \) along the z-axis, we interpret:

- \(s\mathbf{i} \) as a scalar multiple of \(\mathbf{i} \), i.e., \(s \) times the x-component.
- The term \(\mathbf{i} \) as adding 1 to the x-component.
- The term \(- \mathbf{k} \) as subtracting 1 from the z-component.

Thus, combining these:

\[
\mathbf{U} = (s + 1)\mathbf{i} + 0\mathbf{j} - 1\mathbf{k}
\]

Expressed as components:

\[
\mathbf{U} = (s + 1, \ 0, \ -1)
\]

Deciphering Vector V

The vector \(\mathbf{V} \) is explicitly given as:

\[
\mathbf{V} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}
\]

which translates directly into components:

\[
\mathbf{V} = (2, -1, 3)
\]

Mathematical Framework for Calculating Area

The core idea behind calculating the area of the parallelogram formed by two vectors in three-dimensional space involves the cross product. The magnitude of the cross product of the two vectors gives the area of the parallelogram:

$$
\text{Area} = |\mathbf{U} \times \mathbf{V}|
$$

This is a fundamental property in vector calculus, rooted in the geometric interpretation of the cross product. It combines both the directions and magnitudes of the vectors, and the resulting vector is orthogonal (perpendicular) to the plane containing $\mathbf{U}$ and $\mathbf{V}$.

Calculating the Cross Product $\mathbf{U} \times \mathbf{V}$

Given $\mathbf{U} = (s+1, 0, -1)$ and $\mathbf{V} = (2, -1, 3)$, the cross product is computed via the determinant:

$$
\mathbf{U} \times \mathbf{V} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
s+1 & 0 & -1 \\
2 & -1 & 3
\end{vmatrix}
$$

Expanding this determinant:

$$
\mathbf{U} \times \mathbf{V} = \mathbf{i} \cdot
\begin{vmatrix}
0 & -1 \\
-1 & 3
\end{vmatrix}
- \mathbf{j} \cdot
\begin{vmatrix}
s+1 & -1 \\
2 & 3
\end{vmatrix}
+ \mathbf{k} \cdot
\begin{vmatrix}
s+1 & 0 \\
2 & -1
\end{vmatrix}
$$

Let's compute each minor:

1. For $\mathbf{i}$:

$$
(0)(3) - (-1)(-1) = 0 - 1 = -1
$$

2. For $\mathbf{j}$:

$$
(s+1)(3) - (-1)(2) = 3(s+1) + 2 = 3s + 3 + 2 = 3s + 5
$$

3. For $\mathbf{k}$:

$$
(s+1)(-1) - (0)(2) = - (s+1) - 0 = -s - 1
$$

Putting it all together:

$$
\mathbf{U} \times \mathbf{V} = (-1)\mathbf{i} - (3s + 5)\mathbf{j} + (-s - 1)\mathbf{k}
$$

Or, in component form:

$$
\mathbf{U} \times \mathbf{V} = (-1, - (3s + 5), -s - 1)
$$

Magnitude of the Cross Product and the Area

The magnitude of the cross product vector gives the area of the parallelogram:

$$
|\mathbf{U} \times \mathbf{V}| = \sqrt{(-1)^2 + [-(3s + 5)]^2 + (-s - 1)^2}
$$

This simplifies to:

$$
|\mathbf{U} \times \mathbf{V}| = \sqrt{1 + (3s + 5)^2 + (s + 1)^2}
$$

Expanding the squares:

$$
(3s + 5)^2 = 9s^2 + 30s + 25
$$
$$
(s + 1)^2 = s^2 + 2s + 1
$$

Thus,

$$
|\mathbf{U} \times \mathbf{V}| = \sqrt{1 + 9s^2 + 30s + 25 + s^2 + 2s + 1}
$$

Combine like terms:

$$
= \sqrt{(9s^2 + s^2) + (30s + 2s) + (1 + 25 + 1)} = \sqrt{10s^2 + 32s + 27}
$$

Final Expression for the Area

The area of the parallelogram, as a function of the parameter s, is:

$$
\boxed{
\text{Area} = \sqrt{10s^2 + 32s + 27}
}
$$

This expression indicates that the area depends on the value of s, which may be specified or could vary depending on the context of the problem.

Special Cases and Interpretation

- Maximum and Minimum Area: The quadratic expression inside the square root, $10s^2 + 32s + 27$, determines how the area varies with s. Its extrema can be found by analyzing its derivative or completing the square.

- When is the area zero?

The area becomes zero if:

$$
10s^2 + 32s + 27 = 0
$$

Solving for s:

$$
s = \frac{-32 \pm \sqrt{32^2 - 4 \times 10 \times 27}}{2 \times 10}
$$

Compute the discriminant:

$$
1024 - 1080 = -56
$$

Since the discriminant is negative, the quadratic has no real roots, meaning the area never becomes zero for real s. The parallelogram always has a positive area, indicating the vectors are never parallel except perhaps in complex space.

Conclusion: Significance and Applications

Calculating the area of a parallelogram formed by vectors in three-dimensional space is fundamental in physics, engineering, computer graphics, and more. It helps quantify the "spread" or "extent" of vectors in space, which is crucial for understanding force distributions, object orientations, and spatial relationships.

In this specific problem, the dependence of the area on the parameter \(s \) illustrates how vector modifications influence geometric properties. The derived formula,

\[
\boxed{
\text{Area} = \sqrt{10s^2 + 32s + 27}
}
\]

provides a direct means to evaluate the size of the parallelogram for any given \(s \), highlighting the interplay between algebraic parameters and geometric configurations.

Understanding these concepts not only

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